Vector Potential

$\vec{\nabla} \cdot \vec{B} = 0$ everywhere

since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0$

$\Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \qquad A = \text{vector potential}$

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x')}{|x - x'|} d^3x' + \nabla \psi$$

Vector potential is <u>not</u> unique

$\vec{A}' = \vec{A} + \vec{\nabla}\psi$ gives same $\vec{B}$

Choice of gauge: "<u>Coulomb Gauge</u>"

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$ always

$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

Choose gauge such that $\vec{\nabla} \cdot \vec{A} = 0$

Given $A' \Rightarrow A = A' + \nabla \psi$

$\nabla \cdot A = \nabla \cdot A' + \nabla^2 \psi$

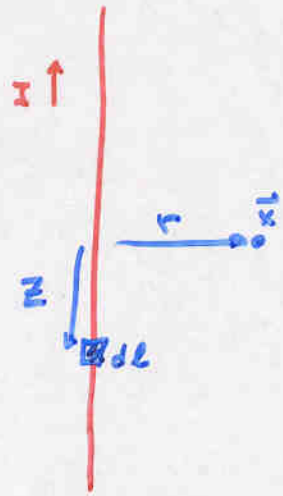
$\nabla \cdot A = 0 \Rightarrow \nabla^2 \psi = -\nabla \cdot A'$

$$\psi = \frac{1}{4\pi} \int \frac{\nabla \cdot A'(x')}{|x - x'|} d^3x'$$

$\Rightarrow \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$

$$\vec{A}(x) = \mu_0 \int \frac{\vec{J}(x')}{|x - x'|} d^3x' + \overrightarrow{\text{constant}}$$

Vector Potential From Long Straight Wire:



$$A_z = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{J_z}{\sqrt{r^2+z^2}}\, d^3x$$

$$= \underline{\underline{\infty}}$$

Try finite length:

$\vec{J} = I\,\delta(x)\,\delta(y)\,\hat{z}$

$$A_z = \frac{\mu_0}{4\pi} \int_{-L}^{L} \frac{I\,\delta(x)\,\delta(y)}{\sqrt{r^2+z^2}}\, d^3x$$

$$= \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{1}{\sqrt{r^2+z^2}}\, dz$$

$$= \frac{\mu_0 I}{4\pi} \log\left[z + \sqrt{r^2+z^2}\right]\Big|_{-L}^{L}$$

$$= \frac{\mu_0 I}{4\pi} \log\left[\frac{\sqrt{r^2+L^2}+L}{\sqrt{r^2+L^2}-L}\right]$$

$$= \frac{\mu_0 I}{4\pi} \log\left[\frac{\sqrt{1+r^2/L^2}+1}{\sqrt{1+r^2/L^2}-1}\right]$$

Lim $L \gg r$

$$A_z = \frac{\mu_0 I}{2\pi}\left[\underline{\underline{\log 2L}} - \log r\right]$$

term $\to \infty$

but doesn't contribute

to curl!

$$\boxed{A_z \simeq -\frac{\mu_0 I}{2\pi} \log r}$$

Vector Potential from long straight wire (cont.)

$$\vec{B} = (0, B_\phi, 0)$$

$$B = \nabla \times A$$

$$\Rightarrow \quad B_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

From Ampre's Law:

$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

In coulomb gauge $\vec{A} \parallel \vec{J} \Rightarrow A_r = 0$

$$-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r} \quad \Rightarrow \quad \boxed{A_z = -\frac{\mu_0 I}{2\pi} \log r}$$

Equally well assume $A_z = 0$

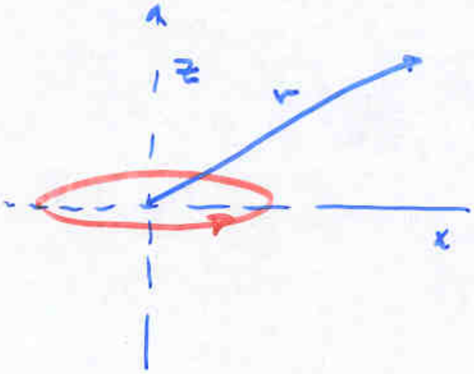
$$\Rightarrow \frac{\partial A_r}{\partial z} = \frac{\mu_0 I}{2\pi r} \quad \Rightarrow \quad A_r = \frac{\mu_0 I z}{2\pi r}$$

$\Rightarrow \quad \vec{A}' = \vec{A} + \nabla \chi$ must be gauge tranform between two.

$$\nabla \chi = \vec{A}' - \vec{A} = \left(\frac{\mu_0 I}{2\pi r} z,\ 0,\ \frac{\mu I}{2\pi} \log r \right)$$

$$\Rightarrow \chi = \frac{\mu_0 I}{2\pi} (\log r)\, z$$

VECTOR POTENTIAL – CURRENT LOOP



assume $\vec{r}$ lies in x-z plane

$$\vec{A}_\phi = A_\phi(x)\,\hat{\phi}$$

$$A_\phi(x) \neq \frac{\mu_0}{4\pi}\int dx^3 \frac{J_\phi(x)}{|x-x'|}$$

For $\vec{r}$ lying in x-z plane

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x(x')}{|x-x'|}\,d^3x' = 0$$

$$A_z = 0 \quad \text{because } J_z = 0$$

$$A_y = \frac{\mu_0}{4\pi}\int \frac{J_y(x')}{|x-x'|}\,d^3x' \qquad J_y(x') = J_{\phi'}\cos\phi' = \frac{I}{a}\,\delta(\cos\theta')\,\delta(r'-a)\cos\phi'$$

$$\boxed{A_y = \frac{\mu_0 I}{4\pi a}\int \frac{a^2\,d\phi'\,\cos\phi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}}}$$

ellipitical integral

$$A_y = \frac{\mu_0 I}{4\pi a}\int d^3x' \frac{\delta(r'-a)\,\delta(\cos\theta')\cos\phi'}{|x-x'|}$$

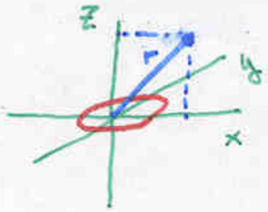
$$= \mathrm{Re}\left\{\frac{\mu_0 I_0}{4\pi a}\int d^3x' \frac{\delta(r'-a)\,\delta(\cos\theta')}{|x-x'|}e^{i\phi'}\right\}$$

$$= \mathrm{Re}\left\{\frac{\mu_0 I}{4\pi a}\int d^3x'\,\delta(r'-a)\,\delta(\cos\theta')\,e^{i\phi'}\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')\cdot Y_{lm}(\theta,\phi)\Big|_{\phi=0}\right\}$$

x, z plane

Vector Potential From a Current Loop.

http://www.physics.uc.edu/~johnson/EM/quarter2/vector.potential/Aloop.nb



Vector potential from loop
r=1 in x,y plane

In[1]:=

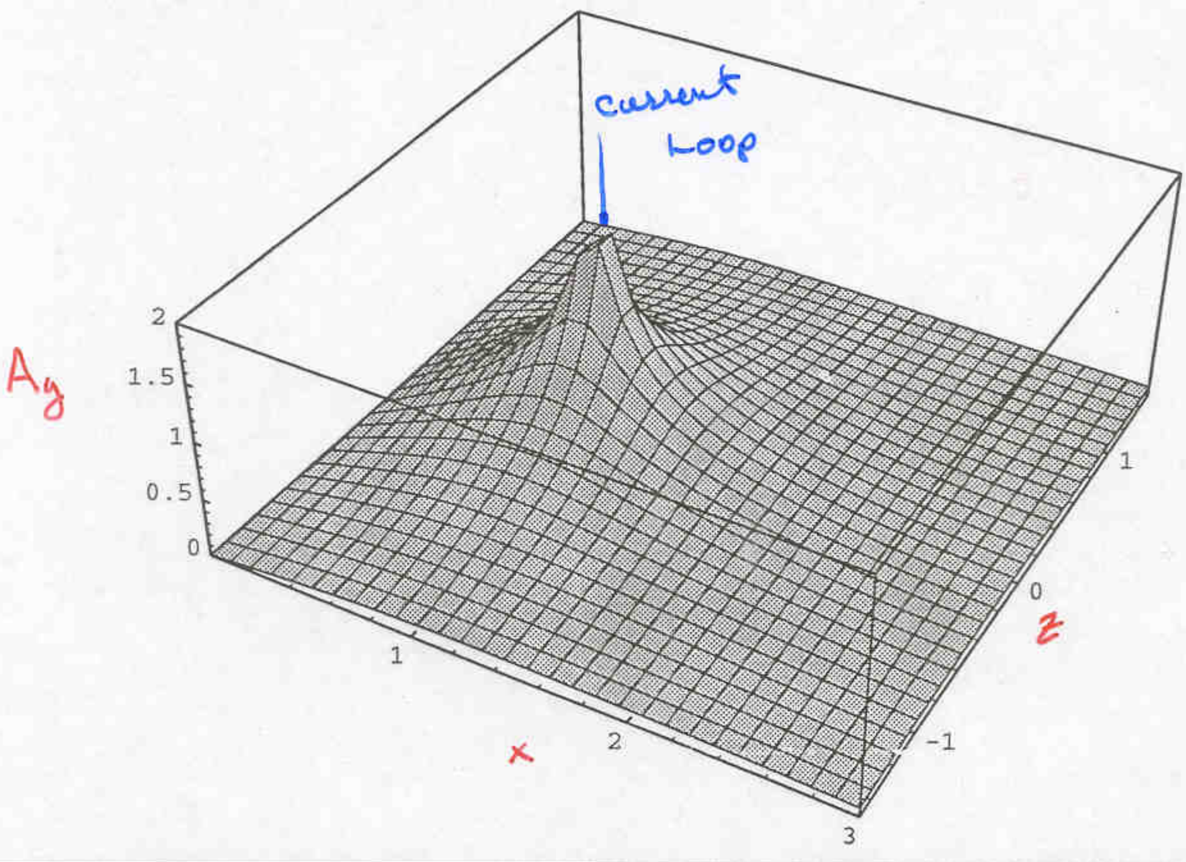
```
Ayact[r_,sint_] = 1/Pi Integrate[Cos[phi]/Sqrt[r^2+1-2 r sint Cos[phi]],
    {phi,0,2 Pi}]
```

Out[1]=

$$\frac{\text{Sqrt}[r^2\ \text{sint}^2]\ \text{Hypergeometric2F1}[\frac{3}{4}, \frac{5}{4}, 2, \frac{4\ r^2\ \text{sint}^2}{(1 + r^2)^2}]}{(1 + r^2)^{3/2}}$$

In[8]:=

```
Plot3D[Ayact[Sqrt[x^2+z^2],x/Sqrt[x^2+z^2]],{x,.001,3},
    {z,-1.5,1.5},PlotPoints->30,PlotRange->{0,2}]
```



$$A_y = \mathrm{Re}\left\{ \frac{\mu_0 I a}{4\pi} \sum_{l,m} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta)\, P_l^m(0) \frac{r_<^l}{r_>^{l+1}} \int_0^{2\pi} d\phi'\, e^{i\phi'} e^{-im\phi'} \right\}$$

$$= \frac{\mu_0 I a}{2} \sum_{l=1}^{\infty} \frac{P_l^1(0) P_l^1(\cos\theta)}{l(l+1)} \; \frac{r_<^l}{r_>^{l+1}}$$

$$P_l^1(0) = 0 \quad \text{for } l = \text{even}$$

$$= (-1)^{(l+1)/2}\, l! / [(l-1)!!]^2$$

$$A_y = \frac{\mu_0 I a}{4} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (2n-1)!!}{2^n (n+1)!} \cdot \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1(\cos\theta)$$

$A_\phi = A_y$ in x-z plane by symmetry!

If $r \gg a \Rightarrow$ keep first term:

$$A_\phi = \frac{\mu_0}{4} I \frac{a^2}{r^2} \sin\theta$$

m = magnetic dipole $= \pi a^2 I$

$$\Rightarrow A_\phi = \frac{\mu_0}{4\pi} \frac{|m| \sin\theta}{r^2} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

$$B = \nabla \times A \Rightarrow B_r = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \sin\theta\, A_\phi$$

$$= \frac{\mu_0 I a^2 \cos\theta}{2 r^3} = \frac{\mu_0}{2\pi} \frac{|m|}{r^3} \cos\theta$$

$$\Rightarrow B_\theta = -\frac{1}{r} \frac{\partial}{\partial r} r A_\phi$$

$$= \frac{\mu_0}{4} \frac{I a^2 \sin\theta}{r^3} = \frac{\mu_0}{4\pi} \frac{|m|}{r^3} \sin\theta$$

Spherical Harmonic Expansion

$r > 1$

```
Aygreater[r_,cost_] =
    -(Sum[(-1)^n (2 n - 1)!!/2^n/(n+1)! /r^(2n+2) *
        LegendreP[2 n+1,1,cost],{n,1,35}]+
        LegendreP[1,1,cost]/r^2);
```

$r < 1$

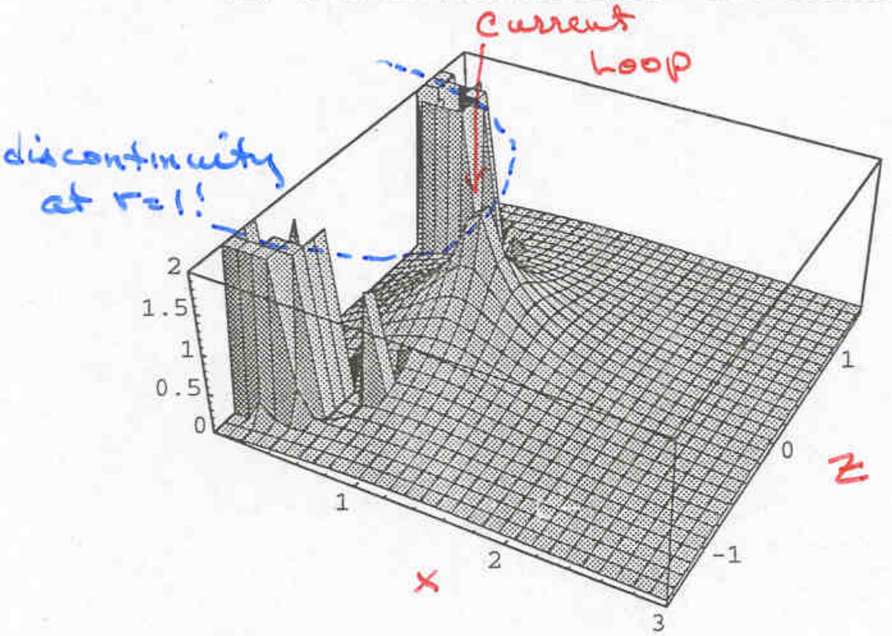
```
Ayless[r_,cost_]=
    -(Sum[(-1)^n (2 n - 1)!!/2^n/(n+1)! r^(2n+1) *
        LegendreP[2 n+1,1,cost],{n,1,35}]+
        LegendreP[1,1,cost] r );
```

Sum first 35 terms.

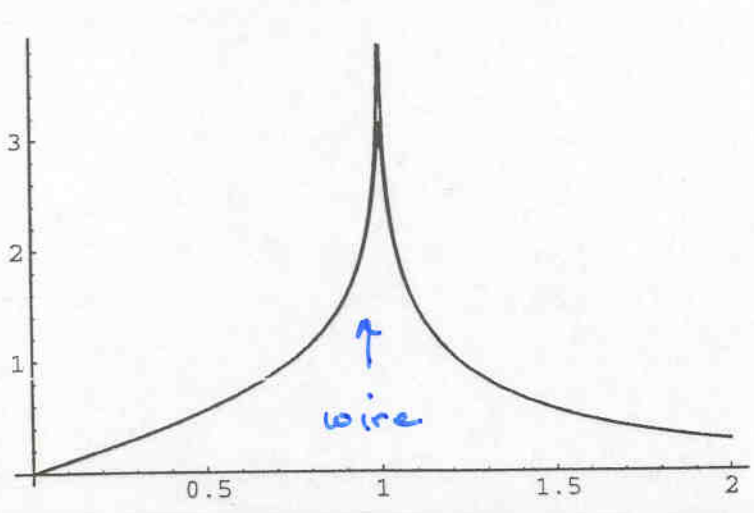
```
Ay[r_,cost_] = If[r>1,Aygreater[r,cost],Ayless[r,cost]];
```

In[10]:=

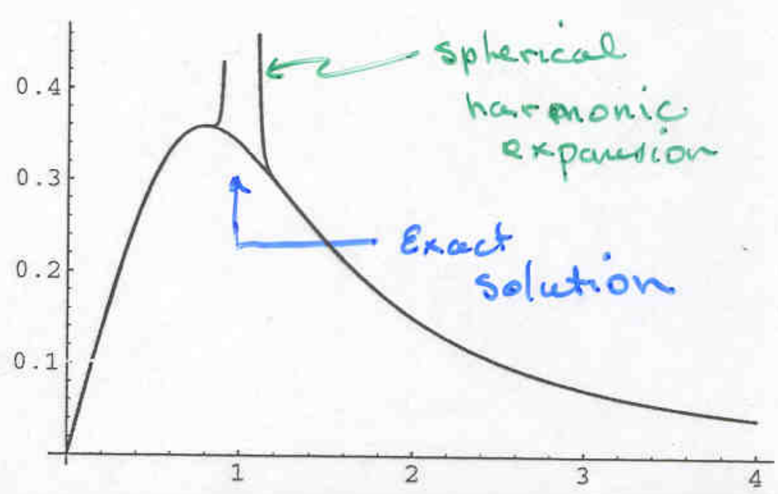
```
Plot3D[Ay[Sqrt[x^2+z^2],z/Sqrt[x^2+z^2]],{x,.001,3},
    {z,-1.5,1.5},PlotPoints->30,PlotRange->{0,2}]
```



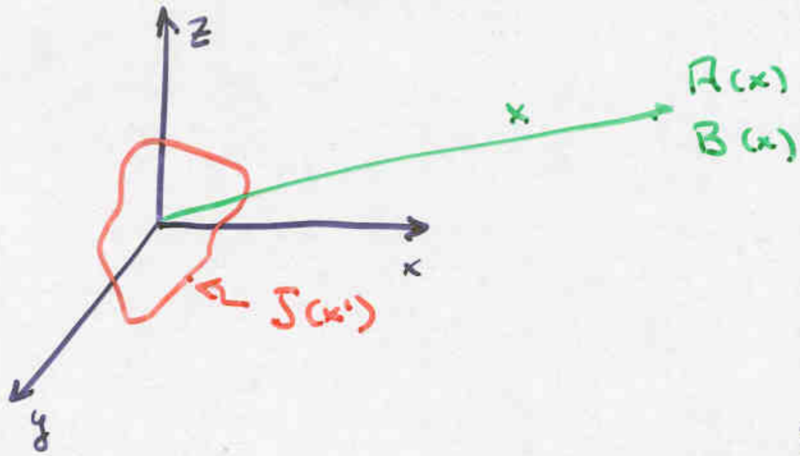
$\theta = \pi$



$\theta = \pi/2$



FIELD FROM LOCALIZED CURRENT DISTRIBUTION



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\overrightarrow{J(x')}}{|x-x'|}\, d^3x'$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

usual expansion $\frac{1}{|x-x'|} = \sum_{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} \sum_{m} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$ doesn't help much.

$$\frac{1}{|x-x'|} = \frac{1}{|x|} + \frac{x\cdot x'}{|x|^3} + \cdots$$

$$\vec{A}(x) = \frac{1}{4\pi}\frac{\mu_0}{|x|}\int \vec{J}(x')\, d^3x' + \frac{\mu_0}{4\pi|x|^3}\int \vec{x}\cdot\vec{x}'\; \vec{J}(x')\, d^3x + \cdots$$

monopole term + dipole term + ...

$\overset{\shortparallel}{0}$

no sources or sinks

Theorem:

$$\int f\,\vec{J}\cdot\vec{\nabla}g + g\,\vec{J}\cdot\vec{\nabla}f\; d^3x' = 0$$

$$\vec{\nabla}\cdot\left[(fg)\vec{J}\right] = fg\,\underbrace{\nabla\cdot J}_{=0} + \nabla(fg)\cdot\vec{J}$$

$$= f\,\vec{J}\cdot\vec{\nabla}g + g\,\vec{J}\cdot\vec{\nabla}f$$

$$\Rightarrow \oint fg\,\vec{J}\cdot\vec{dS} = 0 = \int f\,\vec{J}\cdot\vec{\nabla}g + g\,\vec{J}\cdot\vec{\nabla}f\; d^3x \qquad \text{QED}$$

Monopole term:

$$\int \vec{J}(x')\, d^3x' = 0 \quad \Leftarrow \quad \begin{array}{l} f = 1 \\ g = x' \Rightarrow \nabla' g = \hat{x'} \end{array}$$

Dipole term:

$$f = x_i \quad g = x_j \Rightarrow \int x_i' J_j + x_j' J_i \, d^3x' = 0$$

$$\int \vec{J}\, \vec{x}\cdot\vec{x}'\, d^3x' = \sum_j x_j \int J_i(x')\, x_j'\, d^3x'$$

$$= \frac{1}{2}\sum_j x_j \int x_j' J_i + x_j' J_i \, d^3x'$$

$$= -\frac{1}{2}\sum_j x_j \int x_i' J_j - x_j' J_i \, d^3x'$$

$$= -\frac{1}{2}\sum_{j,k} \varepsilon_{ijk}\, x_j \int (\vec{x}' \times \vec{J})_k\, d^3x'$$

$$= -\frac{1}{2}\, \vec{x} \times \int \vec{x}' \times \vec{J}\, d^3x'$$

$$\boxed{\vec{A}_{\text{Dipole}} = -\frac{\mu_0}{8\pi}\frac{\vec{x}}{|x|^3} \times \int \vec{x}' \times \vec{J}\, d^3x'}$$

$\mathcal{M}$ = magnetization
$= \frac{1}{2}\vec{x}' \times \vec{J}(x')$

Define magnetic dipole moment:

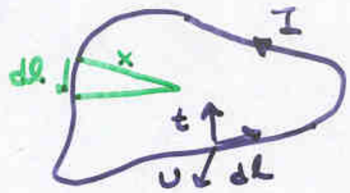
$$\vec{m} = \frac{1}{2}\int \vec{x}' \times \vec{J}(x')\, d^3x' = \int \mathcal{M}\, d^3x'$$

$$\Rightarrow \vec{A}_{\text{Dipole}} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{x}}{|x|^3}$$

$$\nabla \times f\vec{V} = \vec{\nabla} f \times \vec{V} + f\, \vec{\nabla} \times \vec{V}$$

$$= \frac{\mu_0}{4\pi}\, \vec{\nabla} \times \frac{\vec{m}}{|x|}$$

Example: Magnet moment of a loop:



$$J = I\,\delta(t)\,\delta(U)\,\hat{d\ell}$$

$$m = \frac{1}{2}\int \vec{x} \times J(x)\, d^3x'$$

$$= \frac{I}{2}\int \vec{x} \times \vec{d\ell}$$

$$\frac{1}{2}\vec{x}\times\vec{d\ell} = d\vec{A}$$

$$\Rightarrow \vec{m} = I\int d\vec{A} = I\cdot(\text{Area of loop})$$

$$\left[\,|\vec{m}| = I\pi r^2 \quad \text{for current loop}\,\right]$$

Example: Moving Charges:

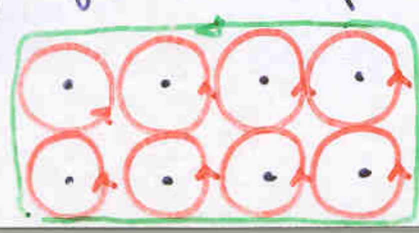
$$\vec{J} = \sum_i q_i \vec{v}_i\, \delta^3(x - x_i)$$

$$\Rightarrow m = \frac{1}{2}\sum_i q_i(\vec{x}_i \times \vec{v}_i)$$

$$= \sum_i \frac{q_i}{2m_i}\vec{L}_i \qquad \vec{L}_i = \text{angular momentum!}$$

if $\frac{q}{m}$ the same: $\Rightarrow \vec{m} = \frac{q}{2m}\sum \vec{L}_i = \frac{q}{2m}\vec{L}$

Example: Array of orbiting charges:



$m = I\cdot\text{Area}$