L^2 function spaces –

$\{F(x)\}$ all functions of 1 parameter

$\Downarrow$ define "norm" via scalar product

$$\int_a^b f^+(\xi)\, g(\xi)\, M(\xi)\, d\xi = (f, g) = \text{number}$$

$$\text{norm} = \|f\| = \sqrt{(f,f)} \quad \text{or} \quad \|f\|^2 = (f,f)$$

properties:

$$(f, g)^+ = (g, f)$$

$$(f, a g_1 + b g_2) = a(f, g_1) + b(f, g_2)$$

Minkowski $\|f + g\| \leq \|f\| + \|g\|$

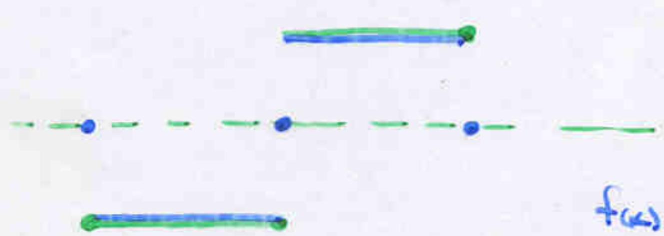
$\Downarrow$ Schwarz $\|(f, g)\| \leq \|f\| \cdot \|g\|$

$\{\text{normalizable functions } f(x)\} \equiv L^2$ space.

Define <u>convergence</u> <u>in</u> <u>mean</u>

$$\{f_n\} \underset{\text{in mean}}{\longrightarrow} f \quad \text{iff} \quad \lim_{n \to \infty} \|f_n - f\| = 0$$

f(x)



$\lim\limits_{n \to \infty} f_n(x)$

$f(x) \neq \lim\limits_{n \to \infty} f_n(x)$ for a countable # of points.

L^2 space (Normed vector space $\int |f|^2 dx \geq 0$)

Define orthonormal sequence of functions as:

$$\{ \varphi_n \mid (\varphi_n, \varphi_m) = \delta_{mn} \}$$

Properties:

1) $\sum_{i=1}^{n} |(f, \varphi_i)|^2 \leq \|f\|^2$

2) Complete means

$$\forall_f \; \exists \{a_n\} = \{(f, \varphi_n)\} \mid \quad \sum_{n=1}^{\infty} a_n \varphi_n \xrightarrow[\text{mean}]{} f$$

3) Must be countably infinite # of φ_n

4) Complete set exists:

Generate via:

$$\varphi_1 = \frac{f_1}{\|f_1\|}$$

$$\varphi_{n+1} = \frac{f_{n+1} - \sum_{i=1}^{n} (f_{n+1}, \varphi_i)}{\|f_{n+1} - \sum_{i=1}^{n} (f_{n+1}, \varphi_i)\|}$$

[5) Closure $\sum \varphi_i^*(x) \varphi_i(x') = \delta(x - x')$]

Eigen values and Eigen vectors:

Given an operator $\mathcal{A}$ that operates on an L^2 space

If:

1) $\mathcal{A}$ is self adjoint ("Hermitian") (i.e. $(f^* \mathcal{A} f)$ is real),

2) $\mathcal{A}$ is positive definite, (i.e. $(f^* \mathcal{A} f) \geq 0$ for all f), and

3) The eigenfunctions of $\mathcal{A}$

$$\mathcal{A} \cdot E_n = a E_n$$

can be found by some variational principle (i.e. minimize $(f^* \mathcal{A} f)/(f^* f)$)

Then:

Eigenvectors form a complete set for vector space.

Examples –

Scalar product: $\int_{-1}^{1} f(x)\, g(x)\, dx$

(1) $f_n(x) = x^{n-1}$ $\qquad n = 1, 2, \cdots$

Orthonormal functions:

$$\left.\begin{aligned} \varphi_1 &= \frac{1}{\sqrt{2}} & \varphi_3 &= \sqrt{\tfrac{5}{2}}\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) \\ \varphi_2 &= \sqrt{\tfrac{3}{2}}\, x & \varphi_4 &= \sqrt{\tfrac{7}{2}}\left(\tfrac{5}{2}x^3 - \tfrac{3}{2}x\right) \end{aligned}\right\} \text{Normalized Legendre polynomials}$$

$$\underbrace{\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right]}_{\text{Hermitian operator}} \varphi_\ell = \underbrace{+\ell(\ell+1)}_{\text{eigen value} \geq 0}\, \varphi_\ell$$

$\Rightarrow$ complete

(2) $f_n(x) = e^{i\pi n x}$ $\qquad n = 0, \pm 1, \pm 2, \cdots$

$$\Rightarrow \varphi_n = \frac{1}{\sqrt{2}}\, e^{in\pi x}$$

$$\left[-\frac{d^2}{dx^2}\right] \varphi_n = +n^2\pi^2 \varphi_n$$

$\Rightarrow$ complete

usually

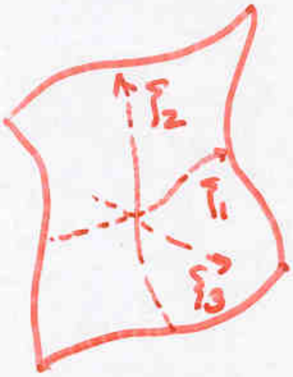
$$\left.\begin{aligned} \varphi'_{n+} &= \frac{\varphi_n + \varphi_{-n}}{\sqrt{2}} = \cos n\pi x \\ \varphi'_{n-} &= \frac{\varphi_n - \varphi_{-n}}{\sqrt{2}\, i} = \sin n\pi x \end{aligned}\right\} \text{Fourier expansion.}$$

Separation of Variables

[treatment: Morse & Feshbach]

Helmholz Eqn.

$$\nabla^2 \phi = -k\phi \text{ subject to}$$



$$\phi_s = f(\xi_1, \xi_2)$$

We have parameterized surface with ξ_1 & ξ_2 = surface variables & $\xi_3 \perp$ surface plus surface $\Rightarrow$ $\xi_3 = x$

$$(ds)^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2$$

$$dV = h_1 h_2 h_3 \, d\xi_1 \, d\xi_2 \, d\xi_3$$

$$\nabla^2 \phi = 0 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \xi_1} \frac{h_1 h_2 h_3}{h_1^2} \frac{\partial \phi}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial \xi_3}\right]$$

Want: - solve one dimensional equations

$$\phi = U_1(\xi_1)\, U_2(\xi_2)\, U_3(\xi_3)$$

Three equations of only one variable:

$$(0) \qquad \frac{1}{f_i(\xi_i)} \frac{\partial}{\partial \xi_i} f_i(\xi_i) \frac{\partial}{\partial \xi_i} U_i(\xi_i) + \left[k_1 \Phi_{1i}(\xi_i) + k_2 \Phi_{2i}(\xi_i) + k_3 \Phi_{3i}(\xi_i)\right] U_i = 0$$

For what coordinate systems can this be done?

Separation of Variables:

Laplace's equation

$$\nabla^2 \phi = 0$$

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\frac{h_1 h_2 h_3}{h_1^2}\frac{\partial}{\partial q_1}\phi + \frac{\partial}{\partial q_2}\frac{h_1 h_2 h_3}{h_2^2}\frac{\partial}{\partial q_2}\phi + \frac{\partial}{\partial q_3}\frac{h_1 h_2 h_3}{h_3^2}\frac{\partial \phi}{\partial q_3}\right] = 0$$

If $\phi = U_1(q_1)\, U_2(q_2)\, U_3(q_3)$

and if: $\underline{h_1 h_2 h_3 = h_i^2\, f_i(q_i)\, g_i(q_j, q_k)}^{*}$

$$\Rightarrow \sum_{i=1}^{3} \frac{1}{h_i^2 f_i U_i(q_i)} \frac{\partial}{\partial q_i} f_i \frac{\partial}{\partial q_i} U_i(q_i) = 0$$

$\{$ can get separation

* very, very special requirement:
most functions won't work.
eg. $h_1 = \log(q_1 \cdot q_2)$

— This is trick of how to separate.

Separation of Variables

$$\nabla^2 \varphi + k\varphi = 0$$

After adding:

$$\sum_{i=1}^{3} \frac{M_i}{S} \frac{1}{f_i} \frac{\partial}{\partial \xi_i} f_i \frac{\partial \varphi}{\partial \xi_i} + k_1 \varphi = 0$$

$\underline{\underline{\text{if}}}$ $$\frac{h_1 h_2 h_3}{h_i^2} = f_1(\xi_1) f_2(\xi_2) f_3(\xi_3) \underline{\underline{M_i}}$$

these are minors of a matrix – very specific form

then: $\dfrac{M_i}{\det S} = \dfrac{1}{h_i^2}$ and $\det S = \dfrac{h_1 h_2 h_3}{f_1(\xi_1) f_2(\xi_2) f_3(\xi_3)}$

and separation <u>works</u>!!

[11 types of coordinate systems + 2 more for Laplace's equation]

<u>Reminder</u>: Solution to separation:

$$\left[\frac{1}{U_i} \frac{1}{f_i} \frac{\partial}{\partial \xi_i} f_i \frac{\partial U_i}{\partial \xi_i} \right] + \left[S \right] \left[k_i \right] = 0$$

f_i parts of Laplace's eqn.

U_i functions of one parameter

k_1 = constant in eqn

$k_2 \to n$ = separation constants

S define by determinants and minors [n^2 elements, $n+1$ constraints] + "more"

Separation of Variables

$$\nabla^2 \phi + k\phi = 0$$

Stäckel determinant:

$$\begin{bmatrix} \Phi_{11}(\xi_1) & \Phi_{21}(\xi_1) & \Phi_{31}(\xi_1) \\ \Phi_{12}(\xi_2) & \Phi_{22}(\xi_2) & \Phi_{32}(\xi_2) \\ \Phi_{13}(\xi_3) & \Phi_{23}(\xi_3) & \Phi_{33}(\xi_3) \end{bmatrix}$$

Define:

$$S = \det[\Phi_{ij}] = \Phi_{11} M_1 + \Phi_{12} M_2 + \Phi_{13} M_3 \qquad (1)$$

↑

minors of determinant.

e.g. $M_1 = \Phi_{22}\Phi_{33} - \Phi_{23}\Phi_{32}$

det[Matrix] = 0 if 2 columns are the same $\Rightarrow$

$$\left.\begin{aligned} 0 &= \Phi_{21} M_1 + \Phi_{22} M_2 + \Phi_{23} M_3 \\ 0 &= \Phi_{31} M_1 + \Phi_{32} M_2 + \Phi_{33} M_3 \end{aligned}\right\} \qquad (2)$$

Take equation (0) for $u_1(\xi_1)$ & multiply by $\frac{M_1}{S} U_2 U_3$

$$\frac{M_1}{S}\frac{1}{f_1}\frac{\partial}{\partial \xi_1} f_1 \frac{\partial \varphi}{\partial \xi_1} + \left[k_1 \underbrace{\frac{M_1}{S}\Phi_{11}}_{\Sigma=1} + k_2 \underbrace{\frac{M_1}{S}\Phi_{21}}_{\Sigma=0} + k_3 \underbrace{\frac{M_1}{S}\Phi_{31}}_{\Sigma=0}\right]\varphi = 0$$

add equations for U_2 and U_3

Cartesian Coordinates:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} + k\varphi = 0$$

$$f_i = 1 \qquad h_i = 1$$

$$S = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial^2 X}{\partial x^2} + kX - k_y X - k_z X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + k_y Y = 0 \qquad \frac{\partial^2 Z}{\partial z^2} + k_z Z = 0$$

Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} + k\varphi = 0$$

$$f_1 = r \qquad f_2 = 1 \qquad f_3 = 1$$

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = 1$$

$$\det S = \frac{h_1 h_2 h_3}{f_1 f_2 f_3} = 1 \qquad M_1 = \frac{S}{h_1^2} = 1 \qquad M_2 = \frac{1}{r^2} \qquad M_3 = 1$$

$$S = \begin{bmatrix} 1 & -1/r^2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial R}{\partial r} + kR - \frac{k_\theta}{r^2} R - k_z R = 0$$

$$\frac{\partial^2 \Theta}{\partial \theta^2} + k_\theta \Theta = 0 \qquad \frac{\partial^2 Z_1}{\partial z^2} + k_z Z_1 = 0$$

Separation of Variables

Example: Spherical:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\varphi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\varphi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} + k\varphi = 0$$

$f_1(r) = r^2$ $\qquad$ $f_2(\theta) = \sin\theta$ $\qquad$ $f_3(\phi) = 1$

$h_1 = 1$ $\qquad$ $h_2 = r$ $\qquad$ $h_3 = r\sin\theta$

$$\det S = \frac{h_1 h_2 h_3}{f_1 f_2 f_3} = \frac{r^2\sin\theta}{r^2\sin\theta} = 1$$

$M_1 = \frac{\det S}{h_1^2} = 1$ $\qquad$ $M_2 = \frac{1}{r^2}$ $\qquad$ $M_3 = \frac{1}{r^2\sin^2\theta}$

$$S = \begin{bmatrix} 1 & -\frac{1}{r^2} & 0 \\ 0 & 1 & -\frac{1}{\sin^2\theta} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} - \frac{k_\theta}{r^2} R = 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta + k_\theta\Theta - \frac{k_\phi}{\sin^2\theta}\Theta = 0$$

$$\frac{\partial^2\Phi}{\partial\phi^2} + k_\phi\Phi = 0$$