Electrodynamics
Final Exam
Spring, '05

Section III
Problem 2

A gauge transformation would take $A^{\mu} \to A'^{\mu} + \partial^{\mu}\chi$. Such at transformation leaves $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ unchanged, but changes $J_{\mu}A^{\mu} \to J_{\mu}A'^{\mu} + J_{\mu}\partial^{\mu}\chi$. So the second term is not gauge invariant. Through current conservation, the gauge transformed second term can be written as $J_{\mu}A'^{\mu} + J_{\mu}\partial^{\mu}\chi = J_{\mu}A'^{\mu} + \partial^{\mu}\left(J_{\mu}\chi\right) - \left(\partial^{\mu}J_{\mu}\right)\chi = J_{\mu}A'^{\mu} + \partial^{\mu}\left(J_{\mu}\chi\right)$ and since total derivatives add nothing to the Lagrangian, the new term adds no new physics. To get the equations of motion:

$$\frac{\partial L}{\partial x^{\mu}} = 0$$

$$\frac{\partial L}{\partial A^{\mu}} = J_{\mu}$$

$$\frac{\partial L}{\partial\left(\partial^{\mu}A^{\nu}\right)} = -\frac{F_{\mu\nu}}{\mu_0}$$

(see Jackson, p. 599 for the last equation). This gives

$$\partial^{\mu}F_{\mu\nu} = \mu_0 J_{\nu}.$$

This equation does not change with the gauge transformation because the added term does not contain A^{μ}.