Electrodynamics
Spring, '05
June 6, 2005
Final Exam

Do one and only one problem in each of the following three sections. If you hand in more than one problem per section, please clearly mark which one is to be graded. If there is any confusion about which one is to be graded, then you will get the score of the lower of the two grades.

Section I. *Special Relativity*

Problem 1: A vector potential from a magnetic dipole is given by $\vec{A}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{|r|^2}$. If the direction of $\vec{m}$ is along the x axis in its rest frame and it is moving past an observer at a velocity βc along the observer's z axis, what are the electrostatic and vector potentials that the observer measures as a function of position and time if the observe is located a distance b from the origin along the y axis? Write down the equations that you would use to determine the observer's measurement of electric field and magnetic induction at this point (however, you do not need to solve these equations).

Problem 2: A 10 GeV/c photon (mass = 0) scatters elastically off of a proton (mass = .939 GeV/c^2) at rest and emerges at an angle of 20^o from its initial direction (sin 20^o=.342 and cos 20^o=.940.) Find the following:

1) the energy of the outcoming photon;
2) the direction and energy of the outcoming proton;
3) the total center of mass energy;
4) the center of mass momentum of each particle;
5) the center of mass scattering angle; and
6) the four-momentum transfer (defined as $q^2=\left(p^\mu_{photon,out}-p^\mu_{photon,in}\right)^2$).

Section II. *Particles in electromagnetic fields*

Problem 1: A particle of mass m and charge q starts at rest at the origin in a crossed electromagnetic field. The electric field and magnetic induction are given by $\vec{E}=E_0\hat{x}$ and $\vec{B}=B_0\hat{y}$ respectively. Find the equations of motion for the particle. Solve them. Over time, what are the bounds of the motion of the particle in the x, y, and z directions. Assume that the motion of the particle is non-relativistic, i.e. $|v|\ll c$.

Problem 2:

In a particular inertial frame, there is a uniform magnetic induction along the z axis, $\vec{B}=B_0\hat{z}$. If a particle of mass m and charge q has an initial velocity of $\vec{v}$ that is confined to the x-y plane in that frame, show that the particle's trajectory is a circle. (Note: do not assume that $|v| \ll c$.) Find the radius of that circle. If an observer is moving at a velocity βc along the z axis with respect to the initial inertial frame, she should see the particle moving in a helix with the same center, the same radius and a velocity along the z axis of $-\beta c$. What are the electromagnetic fields in the new frame? Find the transverse velocity $\vec{v}$ necessary in the new frame for the particle to have a helix as a trajectory with z velocity $-\beta c$ and a helix radius that same as above. Is this transverse velocity larger, smaller, or the same as that in the initial frame? Why?

Section III. *Field theory*

Problem 1: In class we took $\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$ to be the Lagrangian density for the electromagnetic field in the source free region. What is(are) the requirement(s) for a Lagrangian such that the underlying fields conserve energy? Does this Lagrangian meet this(these) requirement(s)? Remembering that the Maxwell stress-energy tensor is

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial^{\mu} A^{\rho}\right)} \partial_{\nu} A^{\rho} - g_{\mu\nu} \mathcal{L}$$

show that $T^{00} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$ plus, possibly, some additional terms.

Problem 2: The electromagnetic Lagrangian density in the presence of sources is given by $\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_{\mu} A^{\mu}$. Show whether this Lagrangian is or is not gauge invariant if the current density is a classical current density (not the current density from some quantum field, but rather only the physical charge and current densities). What are the equations of motion for the electromagnetic potential A^{μ} for this Lagrangian? Perform a gauge transformation on the Lagrangian and find the new equations of motion. Are they the same or different?