Electrodynamics
Spring, '06
Problem Set 6

Problem 7
Jackson 12.16

Part (a)

The canonical stress tensor is given in the equation between eq. 12.103 and 12.104. Since there are no more derivatives in the Proca Lagrangian (eq. 12.91 $L = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{m^2}{8\pi} A_\alpha A^\alpha - \frac{1}{c} J_\alpha A^\alpha$), this equation gives the stress tensor for the Proca Lagrangain too. Thus, in the absence of external sources (i.e. the stress from the field itself):

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\beta A^\lambda - g^{\alpha\beta} \left(-\frac{1}{16\pi} F_{\gamma\delta} F^{\gamma\delta} + \frac{m^2}{8\pi} A_\eta A^\eta \right).$$

The second term is completely symmetric in α and β. So we have to play with the first term. The equation of motion for the A field is given by:

$$\partial_\mu \frac{\partial L}{\partial\left(\partial_\mu A_\nu\right)} = \frac{\partial L}{\partial A_\nu}$$

The LHS is the same as for the EM Lagrangian as given in eq. 12.87 ($-\frac{1}{4\pi} \partial^\beta F_{\beta\alpha}$). The RHS becomes $-\frac{1}{c} J_\alpha + \frac{m^2}{4\pi} A_\alpha$ and $\partial^\beta F_{\beta\alpha} = \frac{4\pi}{c} J_\alpha - m^2 A_\alpha$. Now substituting $\partial^\beta A^\lambda = -F^{\lambda\beta} + \partial^\lambda A^\beta$ into the stress energy tensor gives:

$$\begin{aligned} T^{\alpha\beta} &= -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \left(-F^{\lambda\beta} + \partial^\lambda A^\beta \right) - g^{\alpha\beta} \left(-\frac{1}{16\pi} F_{\gamma\delta} F^{\gamma\delta} + \frac{m^2}{8\pi} A_\eta A^\eta \right) \\ &= \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} - \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^\lambda A^\beta - g^{\alpha\beta} \left(-\frac{1}{16\pi} F_{\gamma\delta} F^{\gamma\delta} + \frac{m^2}{8\pi} A_\eta A^\eta \right) \end{aligned}$$

Now the second term is the only one that is not necessarily symmetric. Working on that term (with the equation of motion above) gives:

$$-\frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\partial^{\lambda}A^{\beta}=\frac{1}{4\pi}F^{\lambda\alpha}\partial_{\lambda}A^{\beta}$$
$$=\frac{1}{4\pi}\left(F^{\lambda\alpha}\partial_{\lambda}A^{\beta}+A^{\beta}\partial_{\lambda}F^{\lambda\alpha}+m^2A^{\alpha}A^{\beta}\right)$$
$$=\frac{1}{4\pi}\left(\partial_{\lambda}\left(F^{\lambda\alpha}A^{\beta}\right)+m^2A^{\alpha}A^{\beta}\right)$$

The first term here is a total derivative and does not add to the energy. Thus:

$$\Theta^{\alpha\beta}=-\frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}\left(-F^{\lambda\beta}+\partial^{\lambda}A^{\beta}\right)-g^{\alpha\beta}\left(-\frac{1}{16\pi}F_{\gamma\delta}F^{\gamma\delta}+\frac{m^2}{8\pi}A_{\eta}A^{\eta}\right)$$
$$=\frac{1}{4\pi}g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta}+\frac{m^2}{4\pi}A^{\alpha}A^{\beta}+g^{\alpha\beta}\left(\frac{1}{16\pi}F_{\gamma\delta}F^{\gamma\delta}-\frac{m^2}{8\pi}A_{\eta}A^{\eta}\right)$$
$$=\frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta}+g^{\alpha\beta}\frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}\right)+\frac{m^2}{4\pi}\left(A^{\alpha}A^{\beta}-g^{\alpha\beta}\frac{1}{2}A_{\eta}A^{\eta}\right)$$

as given in the problem.

Part (b)

Following the reasoning of Section 12.10.C we have:

$$\partial_{\alpha}\Theta^{\alpha\beta}=\frac{1}{4\pi}\partial_{\alpha}\left(g^{\alpha\mu}F_{\mu\lambda}F^{\lambda\beta}+g^{\alpha\beta}\frac{1}{4}F_{\gamma\delta}F^{\gamma\delta}\right)+\frac{m^2}{4\pi}\partial_{\alpha}\left(A^{\alpha}A^{\beta}-g^{\alpha\beta}\frac{1}{2}A_{\eta}A^{\eta}\right)$$
$$=\frac{1}{4\pi}\left(\left(\partial^{\mu}F_{\mu\lambda}\right)F^{\lambda\beta}+F_{\mu\lambda}\left(\partial^{\mu}F^{\lambda\beta}\right)+\frac{1}{2}F_{\gamma\delta}\left(\partial^{\beta}F^{\gamma\delta}\right)\right)+\frac{m^2}{4\pi}\left(\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}+A^{\alpha}\left(\partial_{\alpha}A^{\beta}\right)-\left(\partial^{\beta}A_{\eta}\right)A^{\eta}\right)$$
$$=\frac{1}{4\pi}\left(\left(\frac{4\pi}{c}J_{\lambda}-m^2A_{\lambda}\right)F^{\lambda\beta}+F_{\mu\lambda}\left(\partial^{\mu}F^{\lambda\beta}+\frac{1}{2}\partial^{\beta}F^{\mu\lambda}\right)\right)+\frac{m^2}{4\pi}\left(\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}+A^{\alpha}\left(\partial_{\alpha}A^{\beta}\right)-\left(\partial^{\beta}A_{\eta}\right)A^{\eta}\right)$$
$$=\frac{1}{4\pi}\left(\left(\frac{4\pi}{c}J_{\lambda}-m^2A_{\lambda}\right)F^{\lambda\beta}\right)+\frac{m^2}{4\pi}\left(\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}+A_{\alpha}\left(\partial^{\alpha}A^{\beta}\right)-\left(\partial^{\beta}A^{\eta}\right)A_{\eta}\right)$$
$$=\frac{1}{4\pi}\left(\left(\frac{4\pi}{c}J_{\lambda}-m^2A_{\lambda}\right)F^{\lambda\beta}\right)+\frac{m^2}{4\pi}\left(\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}+A_{\alpha}\left(\partial^{\alpha}A^{\beta}\right)-\left(\partial^{\beta}A^{\alpha}\right)A_{\alpha}\right)$$
$$=\frac{1}{4\pi}\left(\left(\frac{4\pi}{c}J_{\lambda}-m^2A_{\lambda}\right)F^{\lambda\beta}\right)+\frac{m^2}{4\pi}\left(\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}+A_{\alpha}F^{\alpha\beta}\right)$$
$$=\frac{1}{4\pi}\left(\frac{4\pi}{c}J_{\lambda}F^{\lambda\beta}\right)+\frac{m^2}{4\pi}\left(\partial_{\alpha}A^{\alpha}\right)A^{\beta}$$
$$=\frac{1}{c}J_{\lambda}F^{\lambda\beta}$$

where the last line assumes the Lorentz gauge $\partial^{\mu} A_{\mu} = 0$.

Part C:

See Mathematica notebook.