Electrodynamics
Spring, '05
Problem Set 2

Qualifying Exam Problem 1

The interaction part of the electromagnetic Lagrangian is

$$L = \frac{1}{2}mv^2 + \frac{q}{c}\vec{v}\bullet\vec{A} - q\Phi$$

Euler's equation is:

$$\frac{d}{dt}\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial x_i}$$

In vector form after the velocity derivatives are done, this becomes:

$$\begin{aligned}
\frac{d\vec{p}}{dt} + \frac{q}{c}\frac{d\vec{A}}{dt} &= -q\vec{\nabla}\Phi + \frac{q}{c}\vec{\nabla}\left(\vec{v}\bullet\vec{A}\right) \\
&= -q\vec{\nabla}\Phi + \frac{q}{c}\left(\left(\vec{v}\bullet\vec{\nabla}\right)\vec{A} + \left(\vec{A}\bullet\vec{\nabla}\right)\vec{v} + \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{v}\right)\right) \\
&= -q\vec{\nabla}\Phi + \frac{q}{c}\left(\left(\vec{v}\bullet\vec{\nabla}\right)\vec{A} + \vec{v}\times\vec{B}\right)
\end{aligned}$$

The second line expanded the gradient as per the hint in the problem. Then, on the third line, all position derivatives of the velocity are zero (position and velocity are independent) and $\vec{B} = \vec{\nabla}\times\vec{A}$. Rewriting the above gives:

$$\begin{aligned}
\frac{d\vec{p}}{dt} &= -q\vec{\nabla}\Phi + \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\left(\vec{v}\bullet\vec{\nabla}\right) - \frac{q}{c}\frac{d\vec{A}}{dt} \\
&= -q\vec{\nabla}\Phi + \frac{q}{c}\vec{v}\times\vec{B} + \frac{q}{c}\left(\vec{v}\bullet\vec{\nabla}\right)\vec{A} - \frac{q}{c}\frac{\partial\vec{A}}{\partial t} - \frac{q}{c}\left(\frac{d\vec{x}}{dt}\bullet\vec{\nabla}\right)\vec{A} \\
&= -q\left(\vec{\nabla}\Phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) + \frac{q}{c}\vec{v}\times\vec{B} \\
&= q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}
\end{aligned}$$