Problem Set 1
Electrodynamics
Winter, '05
Due: January 14, 2005

Problem 1: Calculate the divergence and the curl everywhere (including the origin which may have to be handled separately) in Cartesian coordinates for

$$\vec{V} = \frac{-y}{x^2+y^2}\hat{x} + \frac{x}{x^2+y^2}\hat{y}.$$

As you can see, the curl is zero everywhere except the origin. Even though the curl is not zero everywhere, if you can define a simply connected domain (*i.e.*, a domain with no holes in it) and if the curl of the vector function is zero within that domain, you can define a unique (to within a constant), continuous scalar field over the domain whose gradient is the vector field.

a. If you define the space as all points on the plane except the half line $y = 0$, $x \le 0$, then you have a simply connected domain. Find the scalar field whose gradient is the above vector field by path integration of the above function. Start at the point $\{x, y\} = \{1, -1\}$ and integrate first in the y direction to the general point $\{1, y\}$. Then integrate in the x direction from that point to $\{x, y\}$. This path avoids the half line excluded from the domain. Check your result by showing that the gradient of the function you get is the vector field given above.
b. Now exclude the half line $x = 0$, $y \ge 0$ by starting at the same point and first integrating in the x to $\{x, -1\}$ and then in the y to $\{x, y\}$. Again, check your result by taking its gradient.
c. Where are the two scalar functions that you derived in sections (a) and (b) the same and why? Where are they different?

Problem 2: Separable coordinate systems are very important for solving problems in physics. There are eleven separable three dimensional coordinate systems (see Morse and Feshbach, Chapt. 5) of which Cartesian, cylindrical, and spherical are three. The only other one that I have ever used is prolate spheroidal which is used for solving for the energy levels and the wave functions of the H_2^+ ion. If ξ_1, ξ_2, ξ_3 are the three coordinates, then holding the ξ_1 constant and varying ξ_2, ξ_3 gives an ellipsoid of revolution around the z axis with foci at $z = \pm d$. In this coordinate system, the three coordinates can be written as:

$$\xi_1 = d\cosh\mu = 1/2\left(r_1 + r_2\right)$$
$$\xi_2 = \cos\theta = \left(1/2d\right)\left(r_1 - r_2\right)$$
$$\xi_3 = \cos\phi$$

where d_1 is the distance from the origin to either focus, r_1 and r_2 are the distances from the point of interest to the upper and lower foci respectively, ϕ is the angle from the x axis and μ is a number from zero to infinity. The standard Cartesian coordinates are:

$$x = \xi_3\sqrt{\left(\xi_1^2 - d^2\right)\left(1-\xi_2^2\right)}$$
$$y = \sqrt{\left(\xi_1^2 - d^2\right)\left(1-\xi_2^2\right)\left(1-\xi_3^2\right)}$$
$$z = \xi_1\xi_2$$

In this coordinates system, find the general formulas for the gradient, divergence, curl and the Laplacian. If plus charges are situated at $\{x, y, z\} = \{0, 0, d\}$ and $\{0, 0, -d\}$ what is the electrostatic potential written in these coordinates? Take the gradient of this potential to find the field. Take the curl and show that it is zero.

Problem 3: Jackson 1.5. Also explicitly calculate the curl of the vector field that results from taking the gradient of the potential and show that it is zero everywhere including the origin.

Problem 4: Jackson 1.12

Problem 5: Jackson 1.13