Green's Theorem

General Problem: Find $\phi(\vec{x})$ in a volume τ subject to boundry conditions on surface S.

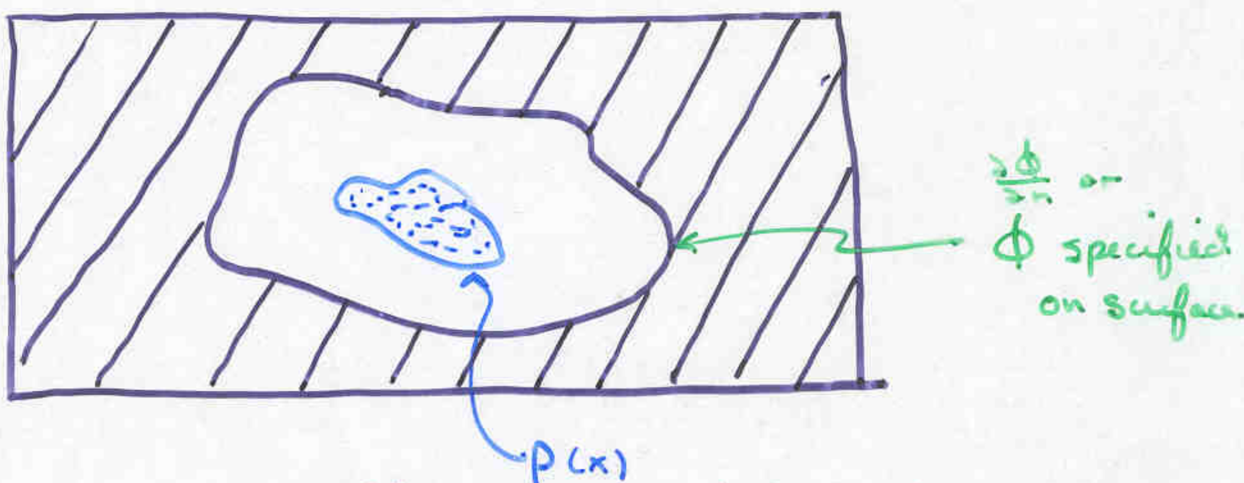
Physical Constraints of ϕ on surface.

Surface may have a charge density σ or a voltage, but only one is arbitrary. One implies the other.

For example:
Surface is a conductor $\Rightarrow$ capacitance of system $\Rightarrow$ $Q \propto V$.

$$\sigma \Leftrightarrow \vec{E}_n \Leftrightarrow \frac{\partial \phi}{\partial n}$$

General Solution



$$\nabla^2 \phi = \frac{\rho}{\varepsilon_0}$$ subject to boundry conditions

Re-write this into an integral equation to explicitly show boundry conditions.

Green's Theorem - Math Interlude

Divergence Theorem $\int_\tau \vec{\nabla}\cdot\vec{V}\,d\tau = \oint_S \hat{n}\cdot\vec{V}\cdot dS$

if $\vec{V} = \xi(x)\,\vec{\nabla}\psi(x)$

then: $\hat{n}\cdot\vec{V} = \xi(x)\,\hat{n}\cdot\nabla\psi = \xi(x)\,\dfrac{\partial\psi(x)}{\partial n}$

$$\vec{\nabla}\cdot\vec{V} = \vec{\nabla}\xi\cdot\vec{\nabla}\psi + \xi\,\nabla^2\psi$$

Green's First Identity

$$\int_\tau\left[\vec{\nabla}\xi\cdot\vec{\nabla}\psi + \xi\nabla^2\psi\right]d\tau = \oint_S \xi\,\frac{\partial\psi}{\partial n}\,dS$$

Switch $\xi(x)$ and $\psi(x)$, do again and subtract:

Green's Theorem [Green's Second Idenity]

$$\int_\tau\left[\xi\nabla^2\psi - \psi\nabla^2\xi\right]dV = \oint_S\left[\xi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\xi}{\partial n}\right]dS$$

Choose: $\xi(\vec{x}') = \phi(\vec{x}')$ and $\psi(x) = \dfrac{1}{|\vec{x}-\vec{x}'|} \equiv \dfrac{1}{R}$

($\vec{x}'$ is the integration variable)

L.H.S $\int_\tau \left[\phi(x')\,\nabla^2\frac{1}{R} - \frac{1}{R}\nabla^2\phi(x')\,d\tau\right.$

$$= \int\left[\phi(x')\left[-4\pi\,\delta^3(x-x')\right] + \frac{\rho}{\varepsilon_0}\frac{1}{R}\right]dV$$

$$= -4\pi\,\phi(x) + \int\frac{1}{\varepsilon_0}\frac{\rho}{R}\,dV$$

Green's Theorem (continued)

$$-4\pi\,\phi(x) + \frac{1}{\varepsilon_0}\int \frac{\rho}{R}\,d\tau = \oint_S \left[\phi(x')\frac{\partial\, 1/R}{\partial n} - \frac{1}{R}\frac{\partial \phi(x')}{\partial n}\right] ds$$

$$\Rightarrow \boxed{\phi(x) = \underbrace{\int_{\tau'} \frac{1}{4\pi\varepsilon_0}\frac{\rho}{R}\,d\tau'}_{\text{part due to charges inside}} + \underbrace{\frac{1}{4\pi}\oint_{S'} ds'\left[\frac{1}{R}\frac{\partial \phi(x')}{\partial n} - \phi(x')\frac{\partial\, 1/R}{\partial n}\right]}_{\text{part due to boundry conditions}}}$$

__Uniqueness__ of ϕ after specifying either ϕ or $\frac{\partial \phi}{\partial n}$ on S

Assume there are two functions ϕ_1 & ϕ_2 which satisfy B.C. & $\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$

$$U = \phi_1 - \phi_2 \quad \Rightarrow \nabla^2 U = 0$$

and either $U(x) = 0$

or $\frac{\partial U}{\partial n} = 0$ on surface.

Green's first idenity $\Rightarrow$

$$\int_\tau d\tau \left[\underset{\overset{\shortparallel}{0}}{U\nabla^2 U} + |\nabla U|^2\right] = \oint_S ds\, \underbrace{U\frac{\partial U}{\partial n}}_{=0}$$

$$\Rightarrow \int_\tau d\tau\, |\nabla U|^2 = 0$$

$$\Rightarrow \vec{\nabla U} = 0$$

$$U = \text{constant}$$

$$\phi_1 = \phi_2 + \text{constant}$$

Green's Functions

[better choice of $\psi(x)$]

Choose:

$$\psi(x') = G(x,x') = \frac{1}{|x-x'|} + F(x,x')$$

where $\nabla'^2 F(x,x') = 0$

$$\Rightarrow \nabla'^2 G(x,x') = -4\pi\,\delta(x-x')$$

$$\Rightarrow \phi(x) = \int_\tau d\tau\, G(x,x') \frac{\rho(x')}{4\pi\epsilon_0} + \frac{1}{4\pi}\oint_S ds \left[G(x,x')\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]$$

Dirichlet boundry conditions

$G_D(x,x')$ on $S = 0$

$\phi(x)$ specified on surface.

$$\phi(x) = \int_\tau d\tau\, G_D(x,x')\frac{\rho(x')}{4\pi\epsilon_0} - \frac{1}{4\pi}\oint_S \phi(x')\frac{\partial G_D}{\partial n}\,ds$$

G_D exists, and is unique [solve Poisson's equation with Boundry value]

$G_D(x,x') = G_D(x',x)$

Neumann boundry conditions

$$\left.\frac{\partial G_N}{\partial n}\right|_{x' \text{ on } S} = -\frac{4\pi}{S}$$

$$\phi(x) = \langle\phi\rangle_S + \int_\tau d\tau\, \frac{G(x,x')\rho(x')}{4\pi\epsilon_0} + \frac{1}{4\pi}\int_S ds\,\frac{\partial\phi(x')}{\partial n} G_N(x,x')$$

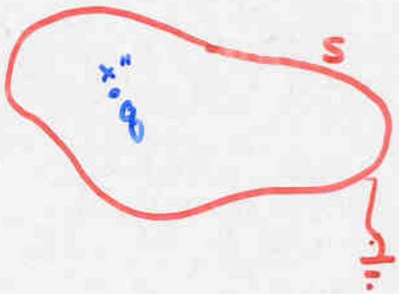
arbitrary constant in potential

$$\langle\phi_S\rangle = \frac{1}{S}\oint_S ds'\,\phi(x')$$

Simple Interpretation –

Green's Functions with Dirichlet Boundary Conditions

$$\left(G(\vec{x},\vec{x}')\Big|_{\vec{x}' \text{ on } S} = 0 \right)$$



Assume $\phi\big|_S = 0$

$$\phi(x) = \int_{\tau'} d\tau' \frac{G(x,x')\,\rho(x')}{4\pi\varepsilon_0} + \frac{1}{4\pi}\oint_{s'} ds' \left[\underbrace{G(x,x')}_{0} \frac{\partial \phi(x')}{\partial n} - \underbrace{\phi}_{0} \frac{\partial G}{\partial n'} \right]$$

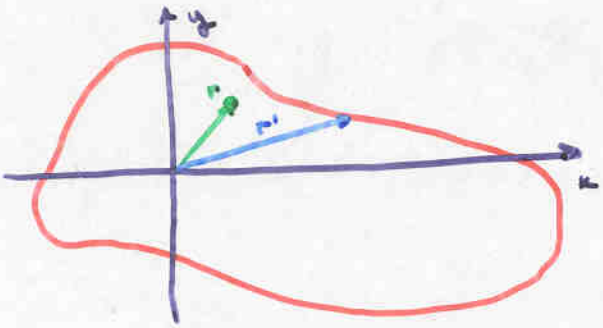
Grounded surface $\qquad \phi(x) = \int_{\tau'} d\tau' \frac{G(x,x')\,\rho(x')}{4\pi\varepsilon_0}$

Now assume 1 charge $\Rightarrow \rho(x') = q\,\delta(x'-x")$

Grounded surface: 1 charge $\qquad \boxed{\phi(x) = G(x,x")\frac{q}{4\pi\varepsilon_0}}$

$\Rightarrow$ If you can solve problem for 1 charge, grounded surface, you can solve for any charge configuration and any voltage boundary conditions.

Green Function
Example



Given: ϕ on surface

$\Rightarrow$ Find green's function for point $\vec{r}$ such that

$$G(r, r') = 0 \Big|_{r' \text{ is on } S.}$$

and $\nabla'^2 G(r, r') = -4\pi\, \delta(r - r')$

$$G(r, r') = \frac{1}{|r - r'|} + F(r, r')$$

such that $\nabla'^2 F = 0$

and $F = -\dfrac{1}{|r - r'|}$ for all r' on S

$\nabla^2 F = 0$ is an averaging operator

$$\lim_{\Delta r \to 0} \frac{\oint_S F(r + \Delta r)\, d^2\Delta r}{S} = F(r)$$

$\Rightarrow$ From boundry conditions can find $F(\vec{r}, \vec{r}')$ for each $\vec{r}$.

$$F(\vec{r}, \vec{r}') \Rightarrow G(\vec{r}, \vec{r}') \Rightarrow \phi(\vec{r})$$

Problem is solved.

Green's Function's In other Dimensions

(1) Divergence Theorem Works in n Dimensions

$$\int d\tau_N \, \vec{\nabla}_n \cdot \vec{V}_n = \oint \vec{V}_n \cdot d\vec{S}_n$$

$\Rightarrow$ n-dimensional Green's Identity

Define:

$$G(x, x') = |x - x'|^{2-n} + F(x, x') \quad \text{if } n \neq 2$$

$$= \log(|x - x'|) + F(x, x') \quad \text{if } n = 2$$

Then $\nabla_n^2 F = 0$

$$\Rightarrow \nabla_n^2 G = S_n \, \delta^n(|x - x'|)$$

S_n = Surface area of unit sphere

$$= \frac{2\pi^{n/2}}{\Gamma(n/2)} = 2, 2\pi, 4\pi, 2\pi^2, \frac{8\pi^2}{3}, \pi^3, \frac{16\pi^3}{15}, \ldots$$

$$\Gamma(1/2) = \sqrt{\pi} \; ; \quad \Gamma(n+1) = n\Gamma(n)$$

$\Rightarrow$ 2-D Green's function:

$$G(x, x') = \overbrace{2\pi \cdot \tfrac{1}{2} \log\left[(x - x')^2 + (y - y')^2\right]}^{\text{Potential from 2D point source = Line of charge.}} + F(x, x'; y, y')$$