Finite Element Solution - Numerical

a) Laplacian is an averaging operator:

Jackson problem 1.10.

$$F(r) = F(0) + \int_0^r \vec{\nabla} F \cdot \vec{dr}$$

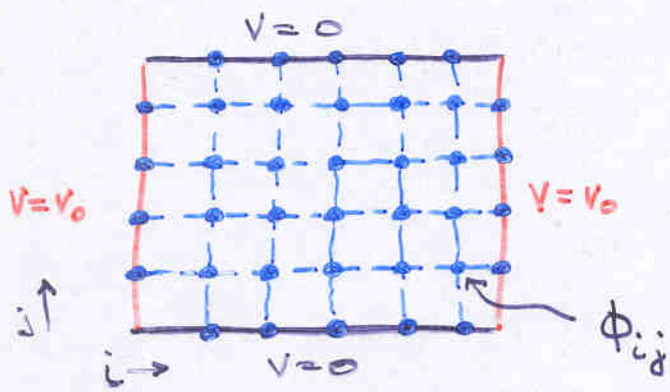
$$\Rightarrow \oint_{\text{sphere}} F(r)\, dA = 4\pi r^2 F(0) + \oint_{\text{sphere}} \int_0^r \vec{\nabla} \cdot F \cdot dr\, dA$$

$$= 4\pi r^2 F(0) + \int_0^r \oint_{\text{sphere}} \vec{\nabla} F \cdot dA\, dr$$

$$= 4\pi r^2 F(0) + \int_0^r \underbrace{\int_{V(r)} \nabla^2 F\, dV}_{=0 \text{ if } \nabla^2 F = 0}\, dr$$

$$\Rightarrow F(0) = \frac{1}{S} \oint_{\text{sphere}} F(r)\, dA$$

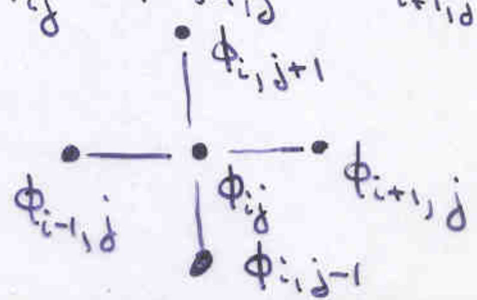
b) Method - Relaxation:



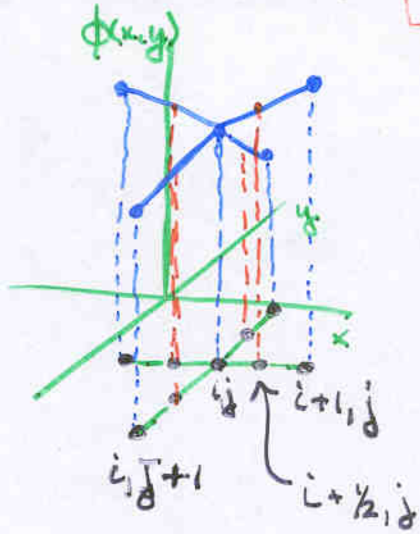
• = grid of points

— = constraint boundary be a given potential.

$$\phi_{ij}^{n+1} = \frac{1}{4}\left(\phi^n_{i-1,j} + \phi^n_{i+1,j} + \phi^n_{i,j-1} + \phi^n_{i,j+1}\right)$$



Laplacian: Discrete approximation



$$\frac{\Delta \phi_{i+1/2,j}}{\Delta x} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$\nabla^2 \phi \simeq \frac{1}{\Delta}\left[\frac{\Delta \phi_{i+1/2,j}}{\Delta} - \frac{\Delta \phi_{i-1/2,j}}{\Delta} + \frac{\Delta \phi_{i,j+1/2}}{\Delta} - \frac{\Delta \phi_{i,j-1/2}}{\Delta} \right]$$

$$\simeq \frac{1}{\Delta^2}\left[\phi_{i+1,j} - \phi_{i,j} - \phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} - \phi_{i,j} - \phi_{i,j} + \phi_{i-1,j} \right]$$

$$\simeq \frac{1}{\Delta^2}\left[\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{ij} \right]$$

$$\text{if } \nabla^2 \phi = 0 \Rightarrow \quad \phi_{ij} = \frac{1}{4}\left[\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} \right]$$

Finite Elements — Theoritical

Real Hilbert Spaces: Two elements:

a) a set of functions $\mathcal{H}$

$$u, v \in \mathcal{H} \Rightarrow u + v \in \mathcal{H}$$

$$u \in \mathcal{H} \Rightarrow a u \in \mathcal{H}$$
$$a \in \mathbb{R}$$

b) a scalar inner product $\langle u | v \rangle$ exists for all $u, v \in \mathcal{H}$

(1) $\langle u | v \rangle = \langle v | u \rangle$

(2) $\langle u | a_1 v_1 + a_2 v_2 \rangle = a_1 \langle u | v_1 \rangle + a_2 \langle u | v_2 \rangle$

(3) $\langle u | u \rangle \geq 0$ for all $u \in \mathcal{H}$

(4) $\langle u | u \rangle = 0$ iff $u = 0$

Solution to Equilibrium Problem $Au = f$

A = differential operator
u = subset of functions of a Hilbert space
f = source function

Properties of A

a) $A(a_1 u_1 + a_2 u_2) = a_1 A u_1 + a_2 A u_2$ linear

b) $\langle Au | v \rangle = \langle u | Av \rangle$ self adjoint

c) $\langle Au | u \rangle \geq 0$ positive

Laplace's Equation

$$\underbrace{-\nabla^2}_{A}\underbrace{\phi}_{U} = \underbrace{\frac{\sigma}{\varepsilon_0}}_{f}$$

(1) Define Inner Product over $\mathbb{D}$ = volume of solution

$$\langle \psi | \phi \rangle \equiv \int_{\mathbb{D}} \psi \phi \, dV = \text{volume integral of product}$$

check properties $\Rightarrow$ OK

(2) Is $-\nabla^2$ self-adjoint?

$$-\langle \psi | \nabla^2 \phi \rangle \stackrel{?}{=} -\langle \nabla^2 \psi | \phi \rangle$$

$$\int \psi \nabla^2 \phi - \phi \nabla^2 \psi \, dV = \oint \psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \, ds$$

Green's Identity

$$= 0$$

if ϕ, ψ satisfy homogeneous boundary conditions

i.e. $\phi, \psi = 0$ or $\frac{\partial \phi}{\partial n}, \frac{\partial \psi}{\partial n} = 0$ on boundary of $\mathbb{D}$

$\Rightarrow$ Admissible functions: $\mathcal{M} \subset \mathcal{H}$

(3) Is $-\nabla^2$ non-negative?

$$\int_{\mathbb{D}} \phi (-\nabla^2) \phi \, dV = \int_{\mathbb{D}} (\nabla \phi)^2 \, dV - \oint \cancel{\phi \frac{\partial \phi}{\partial n}} \, ds$$

0 if $\phi \in \mathcal{M}$

$$\langle \phi | -\nabla^2 \phi \rangle = \langle \nabla \phi | \nabla \phi \rangle \geq 0$$

Convert Equilibrium Problem into minimization

$$I(u) = \langle Au|u\rangle - 2\langle f|u\rangle$$

Theorem 1 Solution $Au_0 = f$ minimizes $I(u)$

Proof:
$$\begin{aligned} I(u) &= \langle Au|u\rangle - 2\langle f|u\rangle \\ &= \langle Au|u\rangle - \langle Au_0|u\rangle - \langle u_0|Au\rangle \\ &\quad + \langle Au_0|u_0\rangle - \langle Au_0|u_0\rangle \\ &= \underbrace{\langle A(u-u_0)|u-u_0\rangle}_{\geq 0} - \langle Au_0|u_0\rangle \end{aligned}$$

minimum when $u = u_0$ QED

Theorem 2: If $Au = f$ has a solution, then it is the function which minimizes $I(u)$.

Proof: Let u_0 be function which minimizes $I(u)$ and let $\bar{u} = u_0 + \varepsilon\eta$

$$f(\varepsilon) \equiv I(u) = I(u_0 + \varepsilon\eta)$$

$$\begin{aligned} 0 = \frac{df}{d\varepsilon} &= \lim_{\varepsilon\to 0} \frac{I(u_0+\varepsilon\eta) - I(u_0)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{2\varepsilon\langle Au_0 - f|\eta\rangle + \varepsilon^2\langle A\eta|\eta\rangle}{\varepsilon} \end{aligned}$$

$$0 = 2\langle Au_0 - f|\eta\rangle$$

$$\Rightarrow Au_0 = f \qquad \text{Q.E.D.}$$

Non Homogeneous boundary Conditions

Constraints:

$$\left.\begin{aligned} B_1 u &= b_1 \\ &\vdots \\ B_n u &= b_n \end{aligned}\right] \text{ Boundary values for surfaces } 1 \to n.$$

$\uparrow$ operators $\qquad$ $\uparrow$ functions on boundary

Let w = some function which satisfies all boundary value equations:

Then define $v = u - w$

$$\Rightarrow B_i v = 0$$

$$Av = f - Aw = f'$$ new equation with homogeneous boundary Conditions

Minimize $I(v) = \langle v | Av \rangle - 2 \langle v | f - Aw \rangle$

$$I(u) = \langle u - w | Au - Aw \rangle - 2 \langle u - w | f - Aw \rangle$$

$$= \langle u | Au \rangle - 2 \langle u | f \rangle - (\langle w | Au \rangle - \langle u | Aw \rangle)$$

$$+ \underbrace{2 \langle w | f \rangle}_{\text{constant}}$$

$$I'(u) = \langle u | Au \rangle - 2 \langle u | f \rangle - \underbrace{(\langle w | Au \rangle - \langle u | Aw \rangle)}_{\text{effects of boundary!}}$$

Specializing to $-\nabla^2$ with Dirichlet Boundary

$u =$ minimizing function
$w =$ function which satisfies B.C.

$$I'(u) = \langle u|A|u\rangle - \left(\langle u|Aw\rangle - \langle w|Au\rangle\right) - 2\langle u|f\rangle$$

$$\langle u|Aw\rangle - \langle w|Au\rangle = -\int u\nabla^2 w - w\nabla^2 u \, dV$$

$$= \oint -u\frac{\partial w}{\partial n} + w\frac{\partial u}{\partial n}\, dS$$

On surface both $u = w = h(s)$

$$\Rightarrow \langle u|Aw\rangle - \langle w|Au\rangle = \oint -\underbrace{h(s)\frac{\partial w}{\partial n}}_{\text{constant}} + h(s)\frac{\partial u}{\partial n}\, dS$$

$$\langle u|Au\rangle = -\int u\nabla^2 u\, dV = \int (\nabla u)^2 dV - \oint u\frac{\partial u}{\partial n}\, dS$$

$$= \int (\nabla u)^2 dV - \oint \underbrace{h(s)\frac{\partial u}{\partial n}}_{\text{Cancels one above.}} dS$$

$$I''(u) = I'(u) + \oint h(s)\frac{\partial w}{\partial n}\, dS$$

$$= \int (\nabla u)^2 dV - 2\int u f\, dV$$

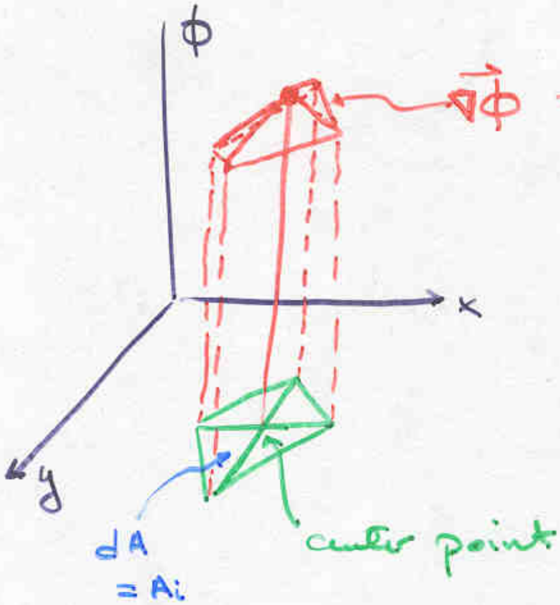
Electrostatics: $u = \phi \quad \nabla u = -E \quad f = \frac{\sigma}{\varepsilon_0}$

$$\Rightarrow I(\phi) = \int (\nabla\phi)^2 dv - 2\int \phi\frac{\sigma}{\varepsilon_0}\, dv$$

if $\sigma = 0 \Rightarrow I(\phi) = \int E^2 dV$

Minimize $\int (\nabla\phi)^2 dV$ on lattice

for a point holding surrounding points fixed



$\vec{\nabla}\phi$ = direction of steepest accent:

$$\vec{\nabla}\phi = \left[\frac{\Delta\phi}{\Delta x}, \frac{\Delta\phi}{\Delta y}\right] \text{ for planes}$$

$$[\nabla\phi]^2 = \left(\frac{\Delta\phi}{\Delta x}\right)^2 + \left(\frac{\Delta\phi}{\Delta y}\right)^2$$

$$I(\phi_{center}) = \int (\nabla\phi)^2 dA = \sum_{areas}\left[\left(\frac{\Delta\phi}{\Delta x}\right)^2 + \left(\frac{\Delta\phi}{\Delta y}\right)^2\right] A_i$$

If square grid $\Rightarrow$

$$I(\phi_0) = \sum_{i=1}^{4} (\phi_0 - \phi_i)^2$$

$$\frac{\partial I}{\partial \phi_0} = 0 = 2\sum_{i=1}^{4} (\phi_0 - \phi_i)$$

$$\Rightarrow \phi_0 = \frac{1}{4}\Sigma\phi_i$$

Very generalizable to other lattice configurations.

Solution – Matrix Method

Minimization $\Rightarrow \quad \Phi_i = \sum_j a_{ij} \Phi_j$

$$[\Phi] = [M][\Phi]$$

$$\left[\begin{array}{c} V \\ \hline \phi \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline B & A \end{array}\right] \left[\begin{array}{c} V \\ \hline \phi \end{array}\right] \quad \begin{array}{l} \text{constraints} \\ \text{values to be found} \end{array}$$

$$[\phi] = [B][V] + [A][\phi]$$

$$\Rightarrow \quad [\phi] = [I - A]^{-1}[B][V]$$

Approximation Method:

$$[\phi]_1 = [A][\phi]_0 + [B][V]$$

$$\vdots$$

$$[\phi]_n = \underbrace{[A]^n[\phi]_0}_{\downarrow \atop 0} + \underbrace{\left(\sum_{i=0}^{n-1} [A]^i\right)}_{\downarrow \atop [I-A]^{-1}}[B][V]$$

How good is the approximation?

$$[\phi]_0 = [\phi_f] + [\delta\phi]$$

$$[\phi_n] = [A]^n[\phi_f] + [A]^n[\delta\phi] + \sum_{i=0}^{n-1}[A]^i[B][V]$$

$$[\phi_f] = [A][\phi_f] + [B][V]$$
$$= [A]^n[\phi_f] + \sum_{i=0}^{n-1}[A]^i[B][V]$$

$$\Delta[\phi]_n = [\phi_n] - [\phi_f] = \underline{\underline{[A]^n}}\,\underline{\underline{[\delta\phi]}}$$

closer starting point $\Rightarrow$ smaller error.

Geometric Series Approximation:

$[A]$ has a set of eigenvalues λ_i and eigenvectors φ_i

$$[\delta\phi] = \sum a_i\varphi_i \quad \text{eigen vector expansion}$$

converges if all $\lambda_i < 1$*

$$\Rightarrow \Delta[\phi]_n = [A]^n[\delta\phi] = \sum a_i\lambda_i^n\varphi_i$$

largest eigenvalue will eventually dominate series.

$$\Rightarrow \Delta[\phi]_n \simeq a_m\lambda_m^n\varphi_m$$

$\Rightarrow$ Geometric sequence.

* largest λ_i associated with longest wave lengths $\Rightarrow$ courser $\rightarrow$ finer grids.