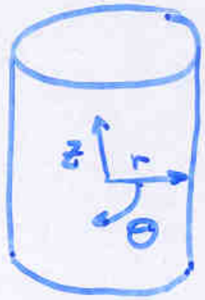
Cylindrical Coordinates



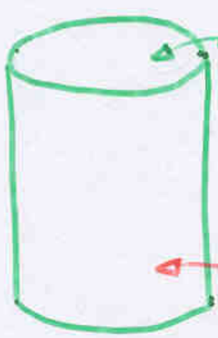
Separation Equations

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

$$\frac{d^2 \Theta}{d\theta^2} + \nu^2 \cancel{\Phi}\,\Theta = 0$$

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} R + \left(k^2 - \frac{\nu^2}{r^2}\right) R = 0$$

Two types of solutions:



$V(Z_1(r,\theta))$ specified

$\Rightarrow$ need a complete set $R(r)\,\Theta(\theta)$

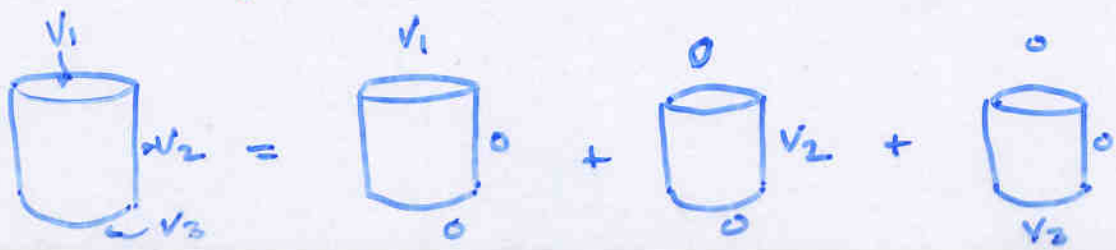
$V(R_1(z,\theta))$ specified

$\Rightarrow$ need complete set $Z(z)\,\Theta(\theta)$

if $k^2 < 0 \Rightarrow \Theta Z = e^{\pm i\nu\theta} e^{\pm ikz}$

complete

if $k^2 > 0 \Rightarrow Z = e^{\pm kz}$ — Not complete



Bessel Equation

Cylindrical Separation: $\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + \left(k^2 - \dfrac{\nu^2}{r^2}\right)R = 0$

Bessel equation: $x = kr$ $\quad \dfrac{d^2R}{dx^2} + \dfrac{1}{x}\dfrac{dR}{dx} + \left(1 - \dfrac{\nu^2}{x^2}\right)R = 0$

Observation: solution must be the same for $\pm\nu$

Try: $R(x) = x^\nu f(x)$

$$\Rightarrow f'' + \frac{2\nu+1}{x} f' + f = 0$$

limits $x \to \infty$ $\qquad$ Behavior

$f'' + f = 0 \Rightarrow$ oscillations

$x \to i\infty$

$-f'' + f = 0 \Rightarrow$ exponential

Recursion relationship:

$$f = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum a_n \left[n(n-1)x^{n-2} + (2\nu+1)n x^{n-2} + x^n \right] = 0$$

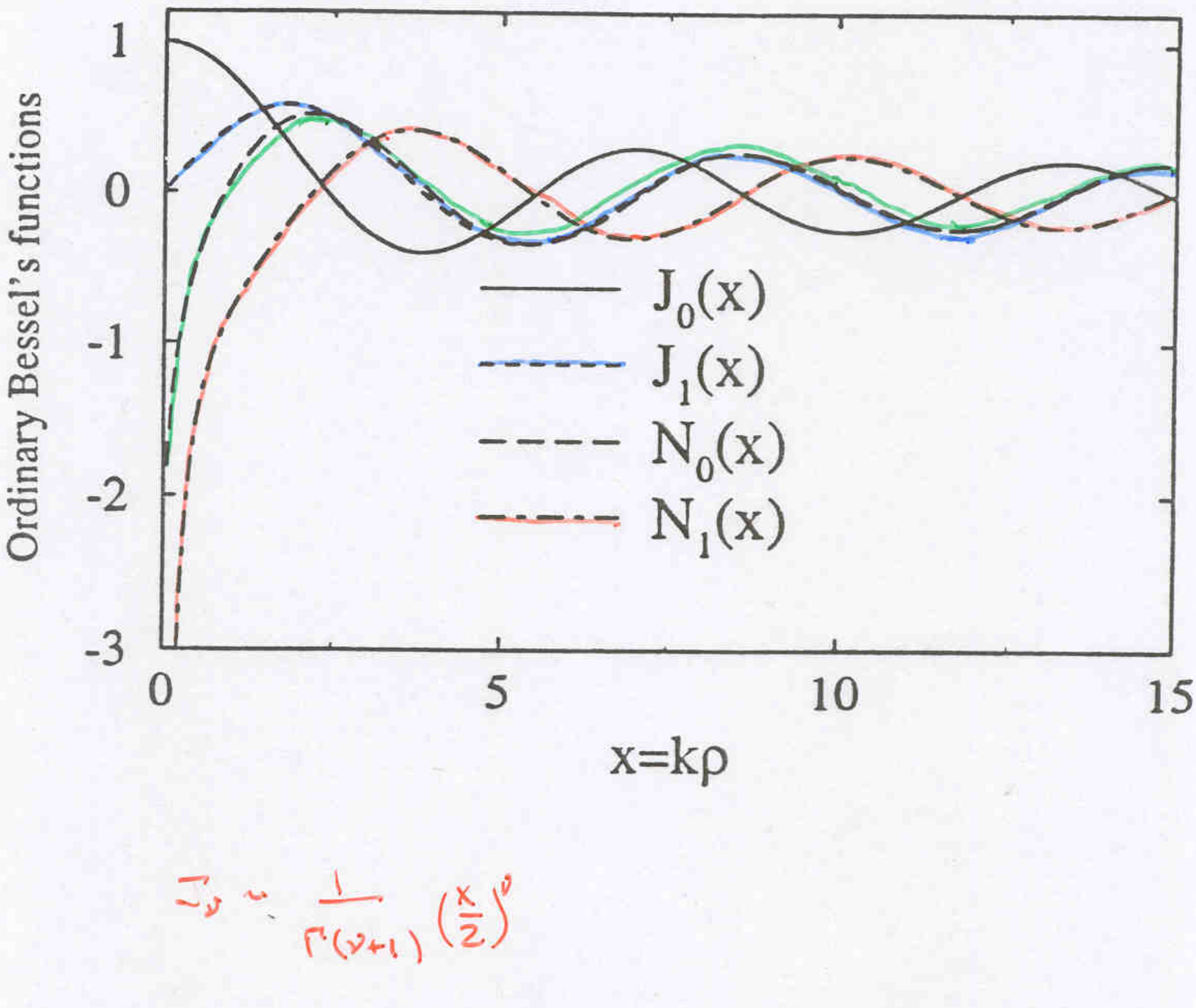
$$\Rightarrow \quad a_{n+2} = \frac{-a_n}{(n+2)(n+2\nu+2)}$$

$j = \frac{n}{2}$

$$a_j = \frac{-a_{j-1}}{2^2 \cdot j\,(j+\nu)} = \frac{(-1)^j\,\Gamma(\nu+1)}{j!\,2^{2j}\,\Gamma(j+\nu+1)}\,\overset{\frac{1}{2^\nu\Gamma(\nu+1)}}{\overset{\parallel}{a_0}} = \frac{1}{2^\nu}\,\frac{(-1)^j}{2^{2j}\,\Gamma(j+\nu+1)}$$

$$J_\nu \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu x}{2} - \frac{\pi}{4}\right)$$

$$N_\nu \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu x}{2} - \frac{\pi}{4}\right)$$



$$J_\nu \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$$

$$N_\nu \sim \frac{2}{\pi} \log\left(\frac{x}{2}\right) + .5772 \qquad \nu = 0$$

$$- \frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu$$

Bessel Functions

Bessel functions of the first kind

$$J_\nu = \left(\frac{x}{2}\right)^{\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

$$J_{-\nu} = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

converges for all x

Problem: ν = integer $\Rightarrow$ $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$

Bessel function of the second kind "Neumann functions"

$\therefore$ Define: $$N_m(x) = \lim_{\nu \to m} \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

$$= \frac{1}{\pi}\left[\frac{d}{d\nu} J_\nu(x) - (-1)^m \frac{d}{d\nu} J_{-\nu}(x)\right]_{\nu=m}$$

Bessel functions of the "third kind" "Hankel functions"

"Symmetric types" of Bessel functions

$$H^{(1)}_\nu(x) = J_\nu(x) + i N_\nu(x)$$
$$H^{(2)}_\nu(x) = J_\nu(x) - i N_\nu(x)$$

Recursion Relationships: Y = any of above

$$Y_{\nu-1}(x) + Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x)$$

$$Y_{\nu-1}(x) - Y_{\nu+1}(x) = 2\,\frac{dY_\nu(x)}{dx}$$

BESSEL FUNCTIONS (CONT.)

Asymtotic behavior

ν Real $\xi > 0$

$x \ll 1$

$$J_\nu \rightarrow \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$$

$$N_0 \rightarrow \frac{2}{\pi} \left[\ln \frac{x}{2} + .5772 \cdots \right] \qquad \nu = 0$$

$$N_\nu \rightarrow -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu \qquad \nu \neq 0$$

$x \rightarrow \infty$

$$J_\nu \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

Imaginary $x = ikr$

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H^{(1)}_\nu(ix)$$

$x \ll 1$

$$I_\nu \rightarrow \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$$

$$K_\nu \rightarrow -\left[\ln \frac{x}{2} + .5772 \cdots \right] \quad \nu = 0$$

$$+ \frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu \qquad \nu \neq 0$$

$x \rightarrow \infty$

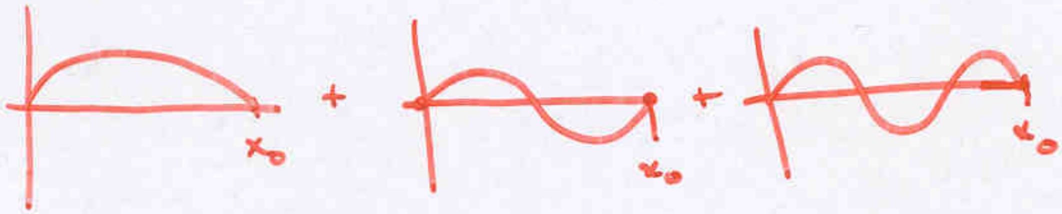
$$I_\nu \rightarrow \frac{1}{\sqrt{2\pi x}} e^x$$

$$K_\nu \rightarrow \sqrt{\frac{\pi}{2x}} e^{-x}$$

Fourier-Bessel Expansion

Complete Set

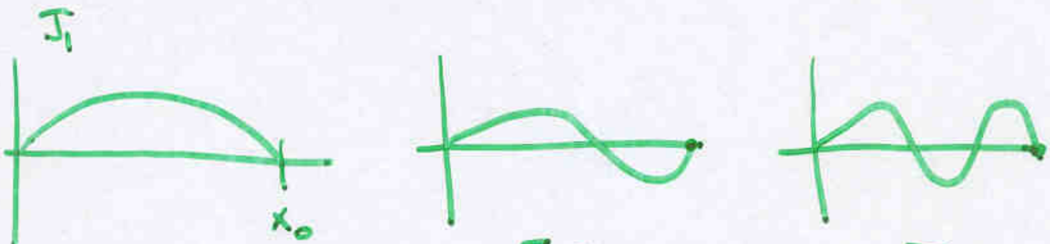
Fourier Expansion



$\sin \frac{\pi}{x_0} x$ $\qquad$ $\sin \frac{2\pi}{x_0} x$ $\qquad$ $\sin \frac{3\pi}{x_0} x$

Expansion in zeros of $\sin \alpha$

Fourier - Bessel



$J_1\left(\frac{x_{11}}{x_0} y\right)$ $\qquad$ $J_1\left(\frac{x_{12}}{x_0} y\right)$ $\qquad$ $J_1\left(\frac{x_{13}}{x_0} y\right)$

$0 \le y \le 1$

Linear independent
[each function has different # of zeros]

Completeness:

$$\sum_{n=1}^{\infty} \frac{J_\nu(r x_{\nu n}/a)\, J_\nu(r' x_{\nu n}/a)}{a^2/2\, [J_{\nu+1}(x_{\nu n})]^2} = \frac{1}{r}\, \delta(r'-r)$$

(complete for measure $r\,dr$)

Orthogonality of Expansion Functions

$$\int_0^{r_0} r\,dr\ J_\nu\!\left(x_{\nu n}\frac{r}{r_0}\right) J_\nu\!\left(x_{\nu m}\frac{r}{r_0}\right) = 0 \text{ if } m\neq n$$

$$\int_0^{r_0} r\,dr\ J_\nu\!\left(x_{\nu m}\frac{r}{r_0}\right)\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} J_\nu\!\left(x_{\nu n}\frac{r}{r_0}\right) + \left(\frac{x_{\nu n}^2}{r_0^2} - \frac{\nu^2}{r_0^2}\right) J_\nu\!\left(\frac{x_{\nu n} r}{r_0}\right)\right] = 0$$

$\Downarrow$ integrate by parts

$$0 = \underbrace{J_\nu\!\left(x_{\nu m}\frac{r}{r_0}\right)}_{0 \text{ at } r_0} \cdot \underbrace{r}_{0 \text{ at zero}} \cdot \frac{d}{dr} J_\nu\!\left(x_{\nu n}\frac{r}{r_0}\right)\Bigg|_0^{r_0} - \int_0^{r_0} dr\ \frac{dJ_\nu\!\left(\frac{x_{\nu m} r}{r_0}\right)}{dr}\, r\, \frac{dJ_\nu(x_{\nu n} r)}{dr}$$

$$+ \int_0^{r_0} r\,dr\ J_\nu\!\left(\frac{x_{\nu m} r}{r_0}\right) J_\nu\!\left(\frac{x_{\nu n} r}{r_0}\right)\left[\frac{x_{\nu n}^2}{r_0^2} - \frac{\nu^2}{r_0^2}\right]$$

$\Downarrow$ same thing for $J_\nu\!\left(\frac{x_{\nu m} r}{r_0}\right)$ and subtract.

$$\underbrace{\frac{(x_{\nu n}^2 - x_{\nu m}^2)}{r_0^2}}_{\text{Since this } \neq 0} \quad \underbrace{\int_0^{r} r\,dr\ J_\nu\!\left(\frac{x_{\nu n} r}{r_0}\right) J_\nu\!\left(\frac{x_{\nu m} r}{r_0}\right)}_{\text{This must be zero.}} = 0$$

Since this $\neq 0$ $\Rightarrow$ This must be zero.

Normalization:

$$\int_0^{r_0} r\,dr\left[J_\nu\!\left(x_{\nu n}\frac{r}{r_0}\right)\right]^2 = \frac{r_0^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2$$

Expansion: $f(r) = \sum_{n=1}^{\infty} A_n J_\nu(x_{\nu n} r/r_0)$

$$A_n = \frac{\int_0^{r_0} r\,dr\ f(r)\, J_\nu(r\, x_{\nu n}/r_0)}{\frac{r_0^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2}$$

FOURIER-BESSEL SERIES – NORMALIZATION

Evaluate: $\int_0^{x_{nm}} J_n^2(x)\, x\, dx$

where $J_n(x_{nm}) = 0$

$J_n(x)$ satisfies the Bessel equation:

$$\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}J_n(x) + \left[1 - \frac{n^2}{x^2}\right]J_n(x) = 0$$

multiply above differential equation by $x^2 \frac{d}{dx}J_n$ and integrate

$$\int_0^{x_{nm}} dx\; x^2 J_n' \left[\frac{1}{x}(xJ_n')' + J_n - \frac{n^2}{x^2}J_n\right] = 0$$

$$\int_0^{x_{nm}} \frac{1}{2}\frac{d}{dx}(xJ_n')^2 + \underbrace{\frac{x^2}{2}\frac{d}{dx}(J_n)^2}_{\substack{\text{integrate}\\ \text{by parts}}} - \frac{n^2}{2}\frac{d}{dx}(J_n)^2 = 0$$

$$\int_0^{x_{nm}} x J_n^2\, dx = \frac{1}{2}x^2(J_n')^2 + \underbrace{\cancel{\frac{x^2}{2}J_n^2}}_{\text{zero at both end points}} - \frac{n^2}{2}J_n^{2\,*}\Bigg|_0^{x_{nm}}$$

* actually zero at endpoints also.

$$xJ_n' = -xJ_{n+1} + nJ_n$$

$$\int_0^{x_{nm}} x J_n^2\, dx = \frac{1}{2}x^2 J_{n+1}^2 - x n \underbrace{\cancel{J_n}}_{=0} J_{n+1}\Bigg|_0^{x_{nm}}$$

$$= \frac{1}{2}x_{nm}^2 J_{n+1}^2(x_{nm})$$

$r = \frac{x}{x_{nm}} r_0 \Rightarrow \int_0^{r_0} r J_n^2\left(\frac{x_{nm} r}{r_0}\right) = \frac{r_0^2}{2} J_{n+1}^2\left(\frac{x_{nm} r}{r_0}\right)$

BESSEL FUNCTIONS (CONT.)

Generating Function:

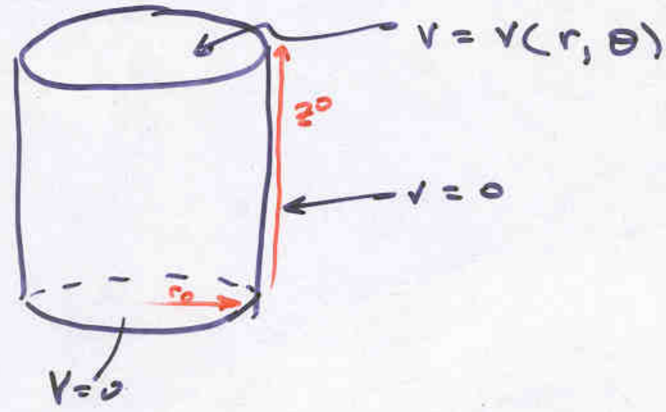
$$e^{z(t^2-1)/2t} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} e^{in\theta} J_n(z)$$

If Y is a type of Bessel function

$$Y_0\left(\sqrt{x^2+y^2-2xy\cos\phi}\right) = \sum_{m=-\infty}^{\infty} Y_m(x) J_m(y)\cos(m\phi)$$

Example I V given on upper surface of Cylinder



$$\Phi_{mk}(r,\theta,z) = e^{im\theta}\left[A_\pm J_m(kr) + B_\pm N_m(kr)\right]\left[e^{-kz} \pm e^{kz}\right]$$

Considerations:

(1) N_m diverge as $r \to 0 \Rightarrow B_\pm = 0$

(2) $A_+ = 0$ because $e^{-kz} + e^{kz}\big|_{z=0} = 2 \neq 0$

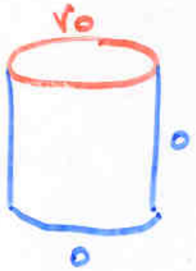
(3) Expand in zeros of $J_m \Rightarrow$ side conditions:

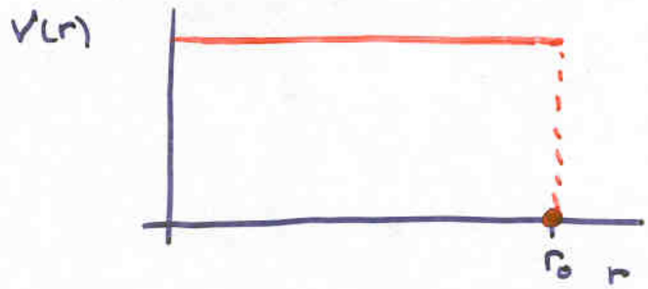
$$\Phi(r,\theta,z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, e^{im\theta} J_m\left(x_{mn}\frac{r}{r_0}\right) \sinh\left(x_{mn}\frac{z}{r_0}\right)$$

$$A_{mn} = \frac{\int_0^{2\pi} d\theta \int_0^{r_0} r\,dr\, e^{-im\phi} J_{|m|}\left(x_{mn}\frac{r}{r_0}\right) V(r,\theta)}{\pi r_0^2 \sinh\left(x_{mn}\frac{z_0}{r_0}\right)\left[J_{|m|+1}(x_{mn})\right]^2}$$

SPECIFIC EXAMPLE

$$V(r,\theta) = \begin{cases} V_0 & r \neq r_0 \\ 0 & r = r_0 \end{cases}$$





(1) A_{mn} with $m \neq 0 = 0$

$\Rightarrow$ only $J_0\left(x_{0n}\frac{r}{r_0}\right)$ in expansion

(2) $x_{0n} \Rightarrow \phi(r_0) = 0$

$$A_{0n} = \frac{2\pi V_0 \int_0^{r_0} r\,dr\, J_0(x_{0n} r/r_0)}{\pi r_0^2 \sinh\left(x_{0n}\frac{z_0}{r_0}\right)\left[J_1(x_{0n})\right]^2}$$

Trick: $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n \frac{dJ_n}{dx} + n x^{n-1} J_n = x^n\left[\frac{dJ_n}{dx} + \frac{n}{x} J_n\right]$

$= x^n J_{n-1}$

Recursion relations

$J_{\nu-1} = \frac{\nu}{x} J_\nu + J_\nu'$

$\Rightarrow x^n J_n(x) = \int dx \frac{d}{dx} x^n J_n = \int dx\, x^n J_{n-1}$

$$\Rightarrow A_{0n} = \frac{2 V_0 \left(\frac{r_0}{x_{0n}}\right)^2 x_{0n} J_1(x_{0n})}{r_0^2 \sinh\left(x_{0n}\frac{z_0}{r_0}\right)\left[J_1(x_{0n})\right]^2}$$

$$= \frac{2V_0}{x_{0n}} \frac{1}{\sinh\left(x_{0n}\frac{z_0}{r_0}\right) J_1(x_{0n})}$$

Cylindrical Geometry – Example 1

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aa=Plot[BesselJ[0,x],{x,0,30},PlotPoints->100]
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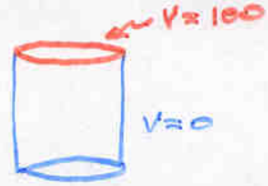
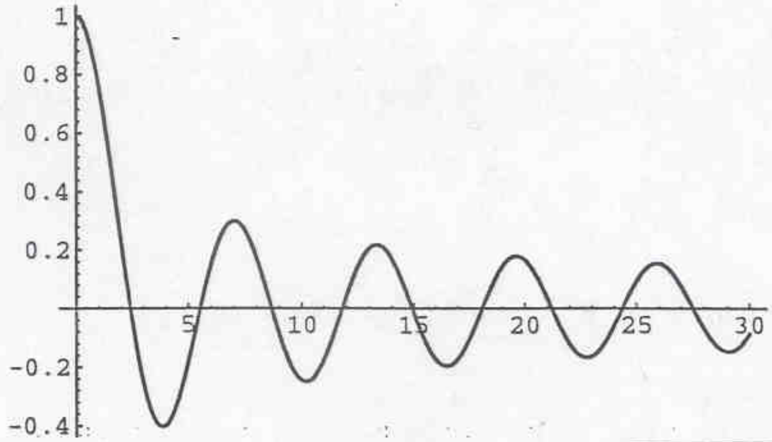
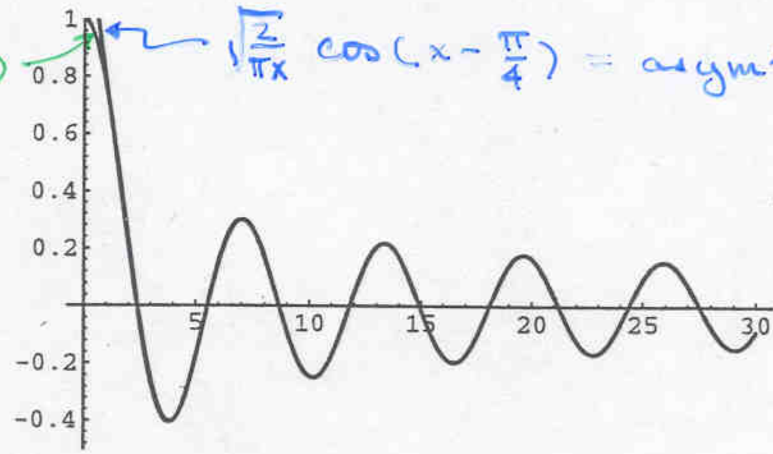


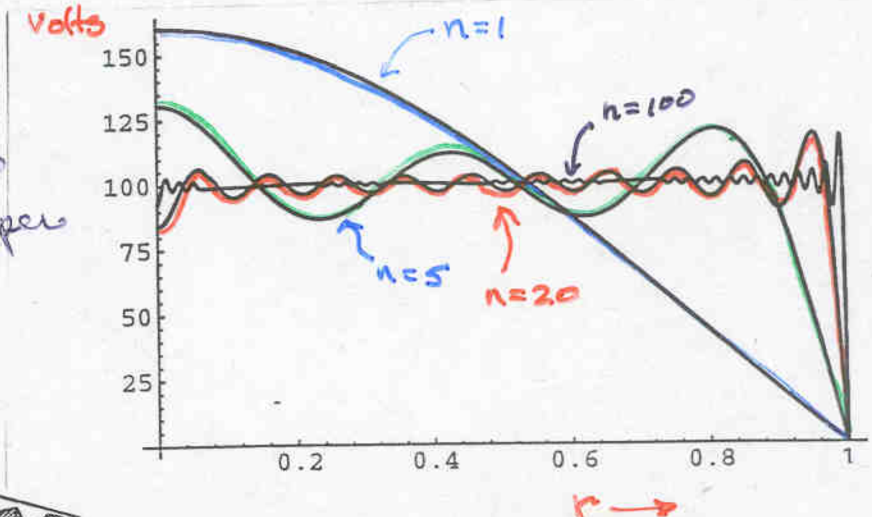


$J_0(x)$

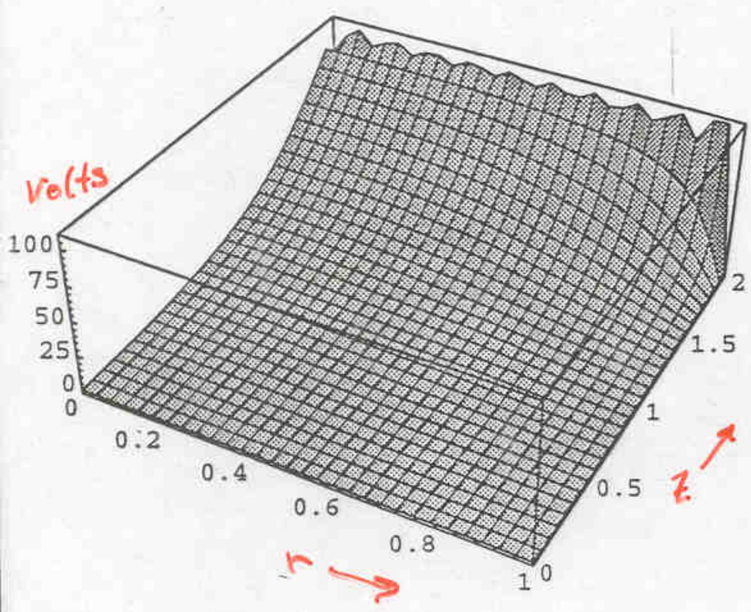


$\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right)$ = asymtotic form

Sum of series up to "n" on upper face



each new $A_n \doteq \frac{V_0}{x_{0n}}$ 1% ⇒ n = 100



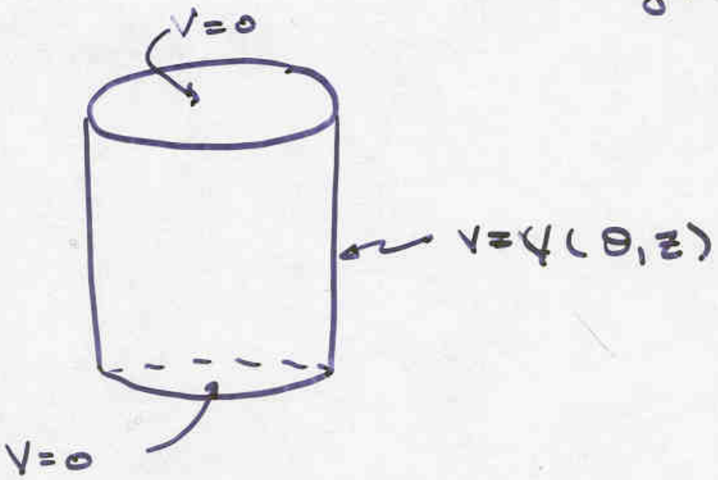
$V_0 = 100$

$r_0 = 1$

$Z_0 = 2$

Voltage in R,Z plane.

EXAMPLE II V specified on side of cylinder



$$\Phi_{km} = e^{im\theta}\left(e^{ikz} \pm e^{-ikz}\right)\left[A_{\pm} I_m(kr) + B_{\pm} K_m(kr)\right]$$

Considerations:

(1) $\cos kz\big|_{z=0} \neq 0 \Rightarrow A_+ \,\&\, B_+ = 0$

(2) K_m's diverge as $r \to 0 \Rightarrow B_{\pm} = 0$

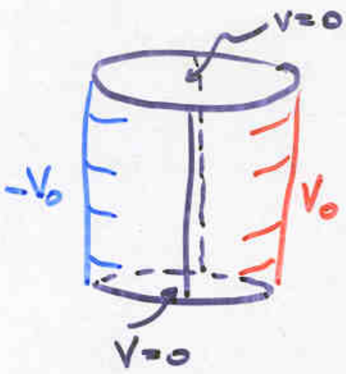
(3) $k = \frac{n\pi}{z_0} \Rightarrow V = 0$ at $z = z_0$

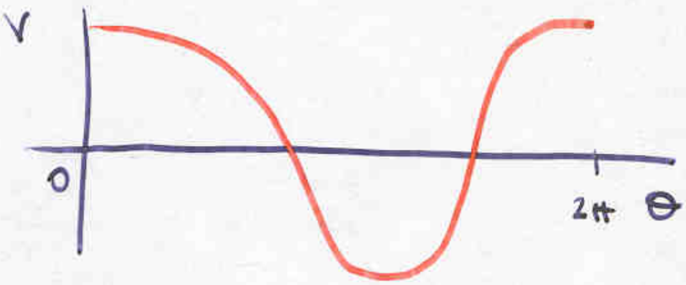
$$\Phi(r,\theta,z) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn}\, e^{im\theta} \sin\left(n\pi \frac{z}{z_0}\right) I_{|m|}\left(\frac{n\pi r}{z_0}\right)$$

where $$A_{mn} = \frac{\int_0^{z_0} dz \int_0^{2\pi} d\theta\, e^{-im\theta} \sin\left(\frac{n\pi z}{z_0}\right) V(\theta,z)}{\pi z_0 I_{|m|}(n\pi r_0/z_0)}$$

Specific Example II

$$V(r_0, \theta, z) = V_0 \cos\theta = V_0/2 \left[e^{i\theta} + e^{-i\theta} \right]$$
$$= 0 \text{ if } z = 0 \text{ or } z_0$$





Considerations:

(1) $A_{mn} = 0$ unless $m = \pm 1$

then $\int d\theta = 2\pi$

(2) $$\int_0^{z_0} dz \sin\frac{n\pi z}{z_0} = \frac{z_0}{n\pi}\left[-\cos\alpha\right]_0^{n\pi} = \frac{2z_0}{n\pi} \text{ if } n = \text{odd}$$
$$= 0 \quad \text{if } n = \text{even}$$

$$\Rightarrow A_{\pm 1\, n\,\text{odd}} = \frac{\frac{V_0}{2} \cdot 2\pi \cdot \frac{2z_0}{n\pi}}{\pi z_0 I_1(n\pi r_0/z_0)} = \frac{2V_0}{n\pi I_1(n\pi r_0/z_0)}$$

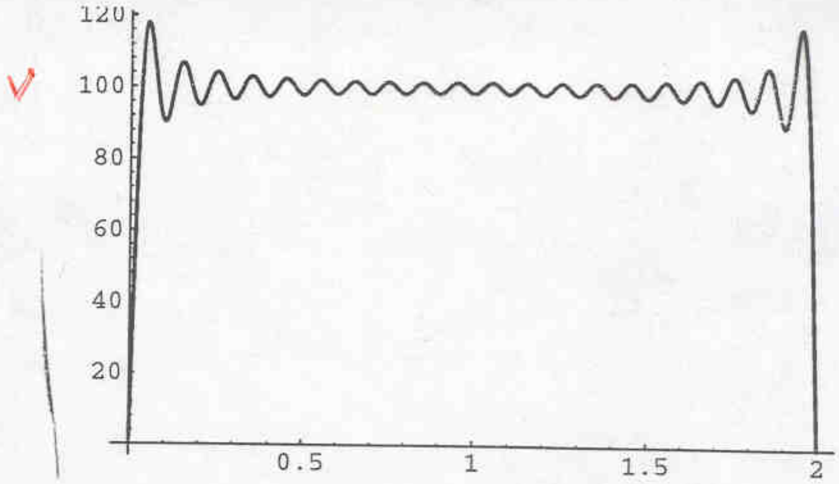
$$\Phi(r, \theta, z) = \frac{4V_0}{\pi}\cos\theta \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{\sin\left(\frac{n\pi z}{z_0}\right)}{n} \frac{I_1\left(\frac{n\pi r}{z_0}\right)}{I_1\left(\frac{n\pi r_0}{z_0}\right)}$$

If $z_0 \to \infty$ and you look at middle $\left(\frac{z}{z_0}\right) = \frac{1}{2}$

$I_n(x) \propto x^n$ as $x \to 0$ $\quad \sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}$ for n odd $\quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$

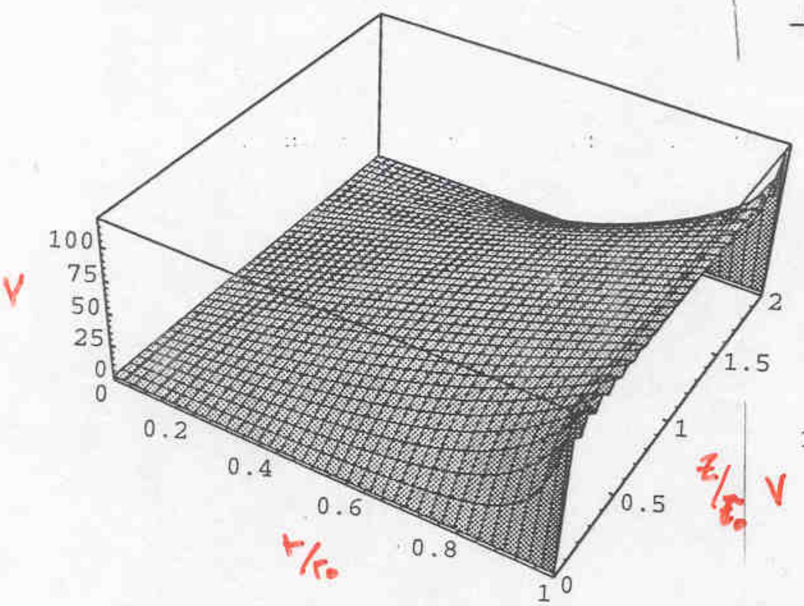
$$\Phi\left(r, \theta, \frac{z_0}{2}\right)_{z_0\to\infty} = V_0 \frac{r}{r_0}\cos\theta$$

Example 2:

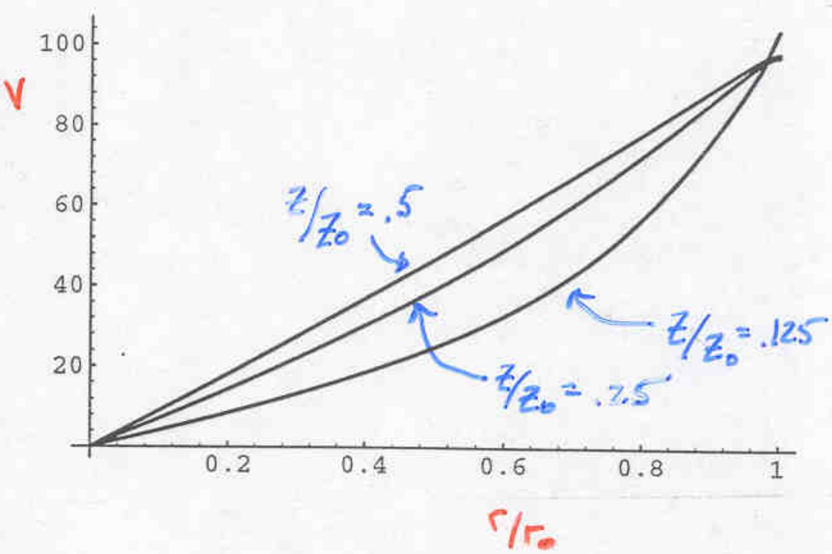


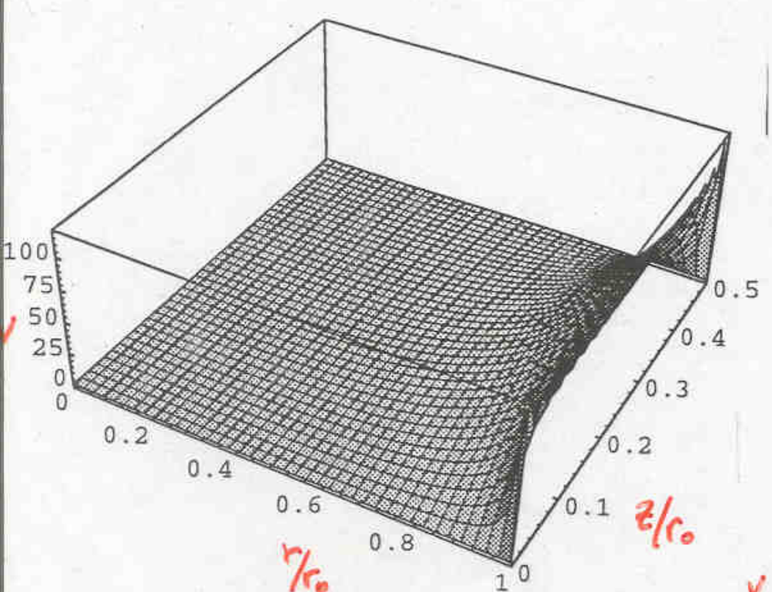
Z @ r = r₀

first 20 terms

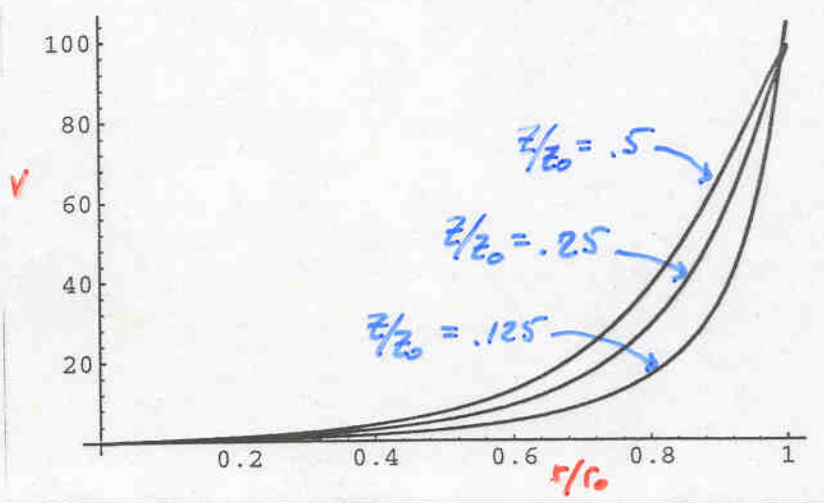


$z_0 = 2r_0$



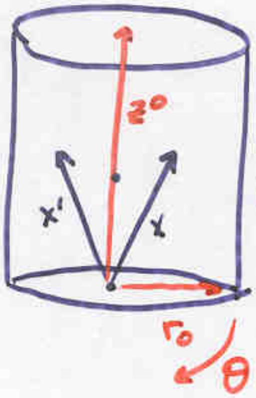


$z_0 = r_0/2$



GREEN'S FUNCTION

(Morse & Feshbach p. 1263)



$$\nabla^2 G(x', x) = -4\pi\, \delta^3(\vec{x} - \vec{x}')$$

$G = 0$ on surface

$$G = \frac{1}{|x - x'|} + F(x, x') \quad \Big|_{\,G = 0 \text{ if } x' \in S}$$

Expand in cylindrical functions:

$$G(x, x') = \sum_{m,n,s} A_{m,n,s}^{(z', r')} \cos[m(\theta - \theta')] \sin\left[\frac{\pi n z}{z_0}\right] J_m\left[\frac{x_{ms} r}{r_0}\right]$$

Note:

Insert G into differential equation:

$$-4\pi\, \delta^3(\vec{x} - \vec{x}') = \sum_{mns} A_{mns} \left[\left(\frac{\pi n}{\ell}\right)^2 + \left(\frac{x_{ms}}{r_0}\right)^2\right] \cos[m(\theta - \theta')] \sin\frac{\pi n z}{z_0} J_m\left(\frac{x_{ms} r}{r_0}\right)$$

Multiply by $\sin\frac{\pi n' z}{z^0} J_m\left(x_{ms'} \frac{r}{r_0}\right)$ and integrate

$$\Rightarrow A_{mns}$$

$$\Rightarrow G(x, x') = \sum_{m,n,s} \frac{\frac{8\epsilon_m}{z_0 r_0^2} \cos[m(\theta - \theta')]}{J_{m+1}^2(x_{ms}) \left[\left(\frac{\pi n}{z_0}\right)^2 + \left(\frac{x_{ms}}{r_0}\right)^2\right]} \times$$

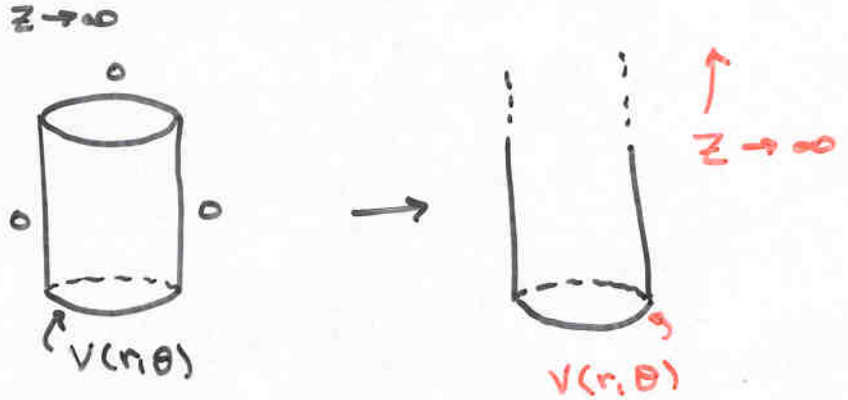
$$\sin\frac{\pi n z}{z_0} \sin\frac{\pi n z'}{z_0} J_m\left(\frac{x_{ms} r}{r_0}\right) J_m\left(\frac{x_{ms} r'}{r_0}\right)$$

$\epsilon_m = 1$ when $m = 0$, $= 2$ otherwise.

Note: should have started with $e^{im\phi}$, then A would have contained an $e^{-im\phi'}$. $\Rightarrow \cos m(\theta - \theta')$. Use this from physical considerations: ie $G(x, x') = G(x', x) \Rightarrow$ only $\cos m(\theta - \theta')$

Extension of Cylindrical Coordinates to Planar Geometries

(1)



$$V(r,\theta,\phi) = \sum_{n,m} \frac{\sinh k_{nm}(L-z)}{\sinh k_{nm} L} J_m(k_{mn}\rho) \times$$

$$\left[A_{nm} \sin m\theta + B_{nm} \cos m\theta\right]$$

Let $L \to \infty$

$$\frac{\sinh k_{nm}(L-z)}{\sinh k_{nm} L} \to e^{-k_{nm} z}$$

(2) $\rho \to \infty$ Same process as going from fourier series to fourier integral:

$$A(x) = \lim_{a\to\infty} \sum_{n=-\infty}^{\infty} A_n e^{i\frac{n\pi x}{a}} = \lim_{\Delta k \to 0} \sum A(k) e^{ikx} \Delta k$$

$$= \int dk\, A(k)\, e^{ikx}$$

$$V(r,\theta,z) \to \sum_m \int dk\, e^{-kz} J_m(k\rho) \times$$

$$\left[A_m(k) \sin m\theta + B_m(k) \cos m\theta\right]$$