Two-Dimensional Case

Complex analysis & Holomorphic Functions

Definition of holomorphic [analytic]

If $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists

and is the same for all complex $\{h_n\} \to 0$
then $f(z)$ is holomorphic and $\lim = \dfrac{df}{dz}$

Cauchy-Riemann Criterion

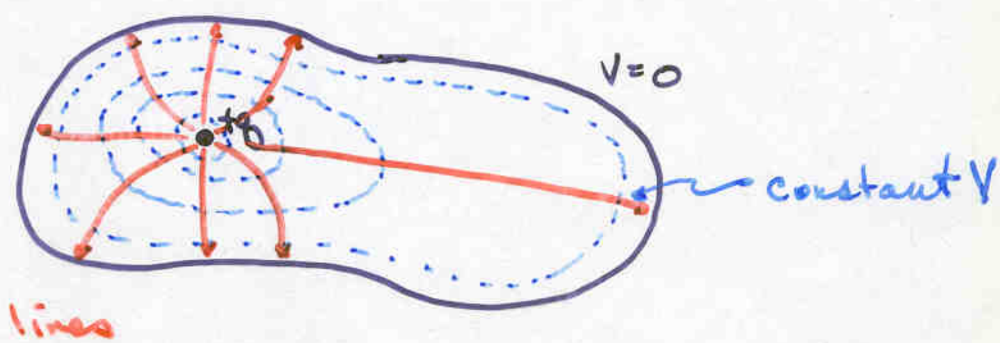
$f(z)$ is holomorphic iff $f(x+iy) = \{U(x,y) + iV(x,y)\}$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \& \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\Rightarrow \quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Both real and imaginary parts
satisfy Laplaces equation

Loci $U = \text{constant} \perp V = \text{constant}$



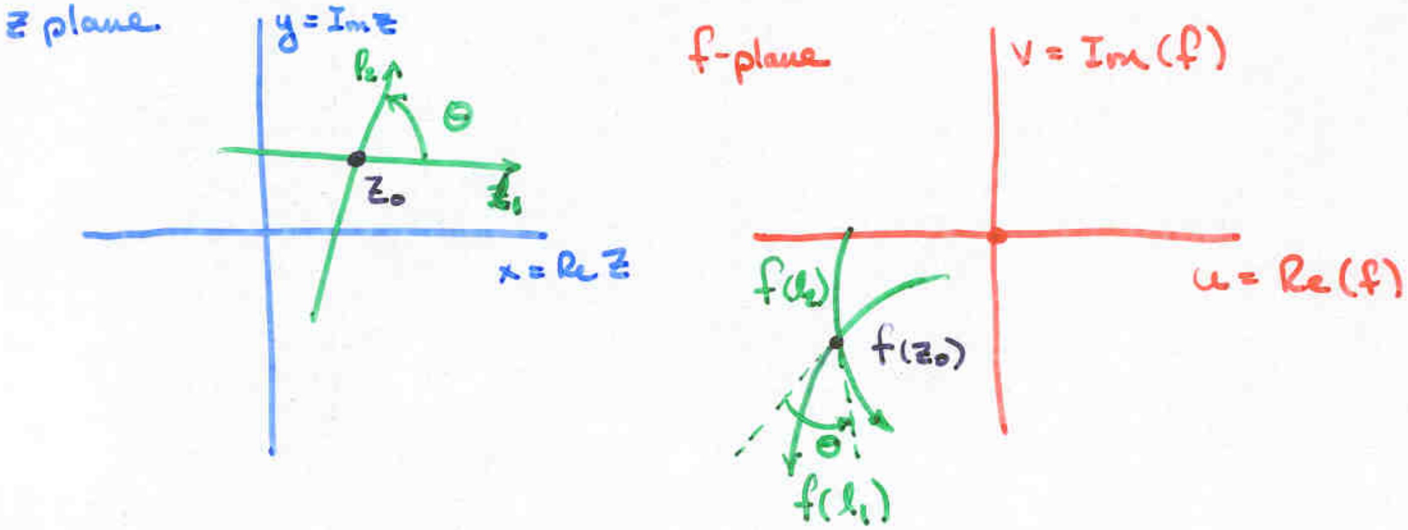
Field lines
= constant U

$\Rightarrow$ 2-dimension Laplaces equation
$\Leftrightarrow$ holomorphic functions.

Conformal Mappings

Def: Conformal mapping $z \to f(z)$ if it preserves sizes of angles.

Theorem: If $f(z)$ is analytic then $z \to f(z)$ is conformal with the sense of the angle preserved.
[transform may include rotation & magnification]



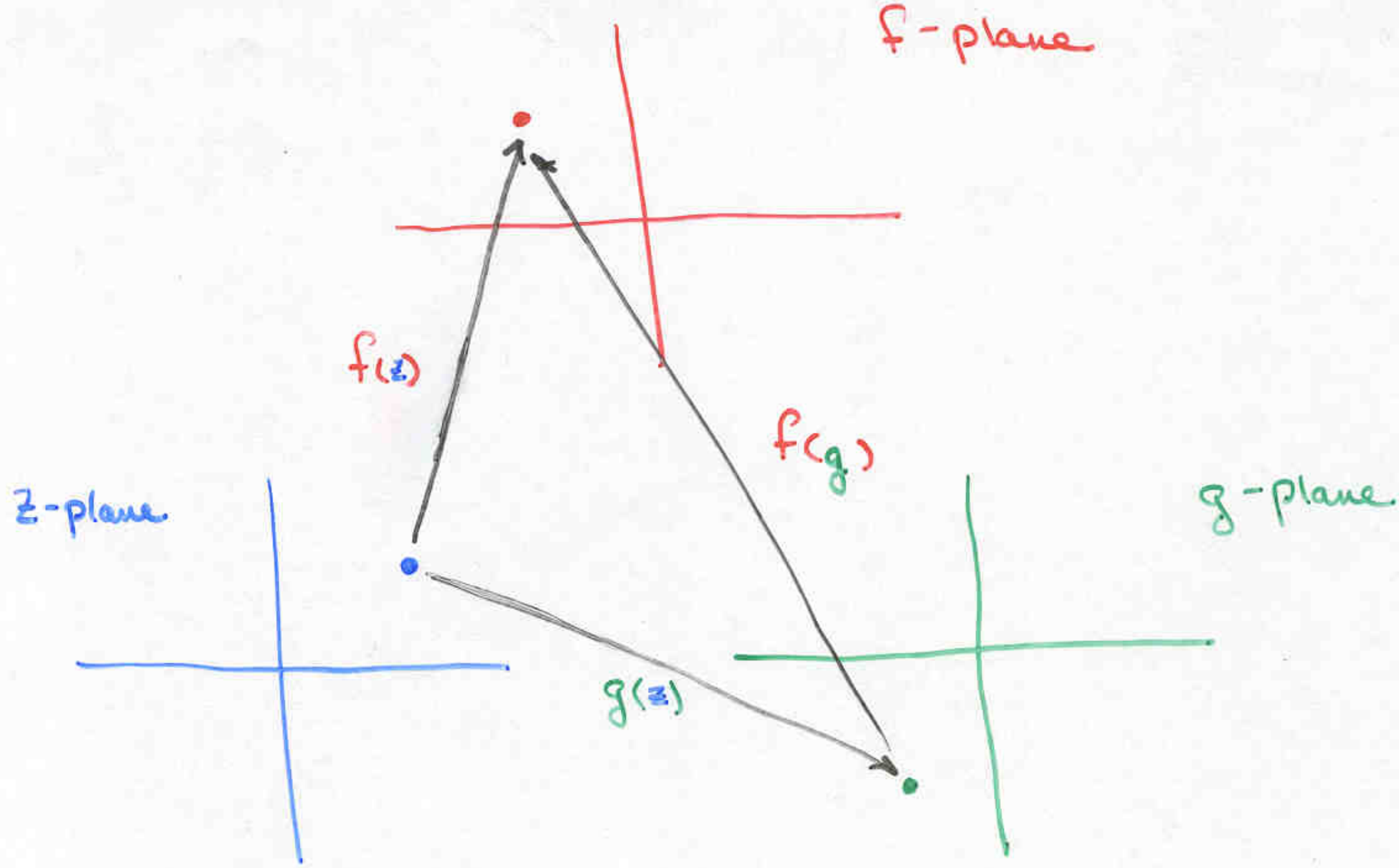
(1) If $f(z)$ is analytic $\Rightarrow$ $Re(f)$ & $Im(f)$ are solutions to laplace's equation

(2) If $z = Z(g)$ is analytic
$\Rightarrow$ conformal mapping &
$f(z) = f(Z(g))$ is analytic in g

(3) $\Rightarrow$ $Re_g(f)$ & $Im_g(f)$ satisfy Laplace's equation.

Two dimensional solution for polygon's, circles, ellipses, etc.

Conformal Mappings



$f(z)$ unknown — to be found

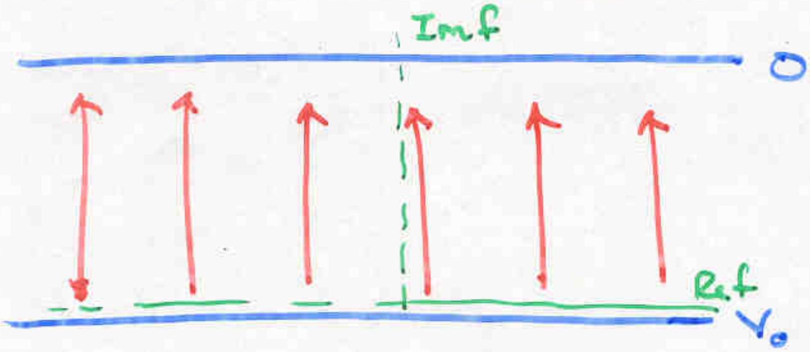
$g(z)$ known — analytic geometry deformation

$f(g)$ known — analytical function of g

$$\Rightarrow \boxed{f(z) = f(g(z))}$$

conformal mapping

Conformal Transformation – Example

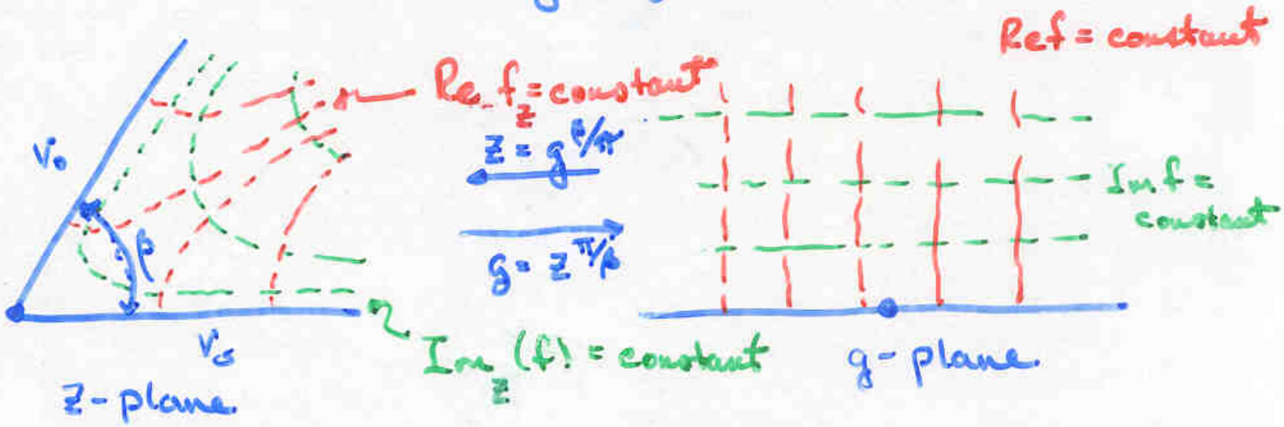


$V = V_0 - E_0 y$

Analytic function

$$f = iV_0 - E_0 z$$

Transform: $z \to Z(g) = g^{\beta/\pi}$

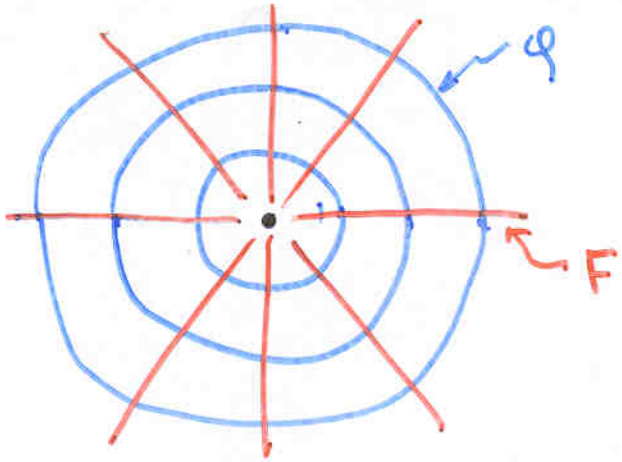


$f = iV_0 - E_0 g$ analytic function in g-plane

$= iV_0 - E_0 z^{\pi/\beta}$ analytic function in z-plane

$$\boxed{\varphi = \mathrm{Im}_z f = V_0 - E_0 r^{\pi/\beta} \sin \frac{\pi}{\beta} \theta}$$

Potential From Single wire



$$\varphi = V_0 \log (x^2+y^2)^{\frac{1}{2}}$$

$$\log(z) = \log r + i\theta$$

$$f(z) = V_0 \log z$$

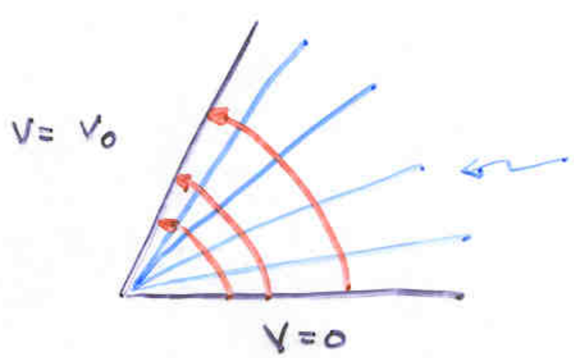
$$\varphi = \mathrm{Re}\left[f(z)\right] = V_0 \log r = V_0 \log (x^2+y^2)^{1/2}$$

Field lines $= F = \mathrm{Im}\left[f(z)\right] = V_0\,\theta = V_0 \arctan y/x$

Branch cut at origin $\Rightarrow$ F increases by $2\pi V_0$ each circuit

Invert problem: $\quad f(z) = V_0 \log z$

$$\varphi = \mathrm{Im}(f(z))$$
$$F = \mathrm{Re}(f(z))$$



$$V = V_0 \cdot \frac{\theta}{\theta_0}$$

$$F = \frac{V_0}{\theta_0} \log r$$

$E_r = 0 \quad E_\theta = \frac{-1}{r}\frac{\partial V}{\partial \theta} = -\frac{1}{r}\frac{V_0}{\theta_0}$

Linear Transform:

$$\frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2} = \frac{w - w_1}{w - w_2} \Big/ \frac{w_3 - w_1}{w_3 - w_2}$$

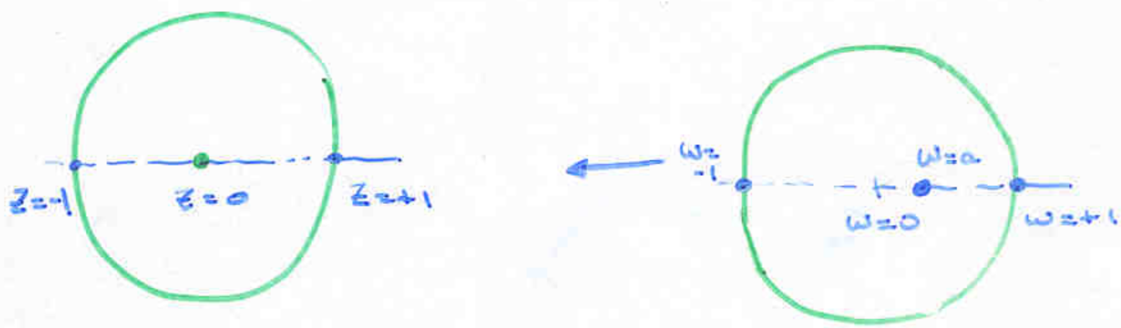
maps: a) 3 points → 3 points

b) circles and lines into circles and lines

Special cases: Rotations

Translations.

Example:



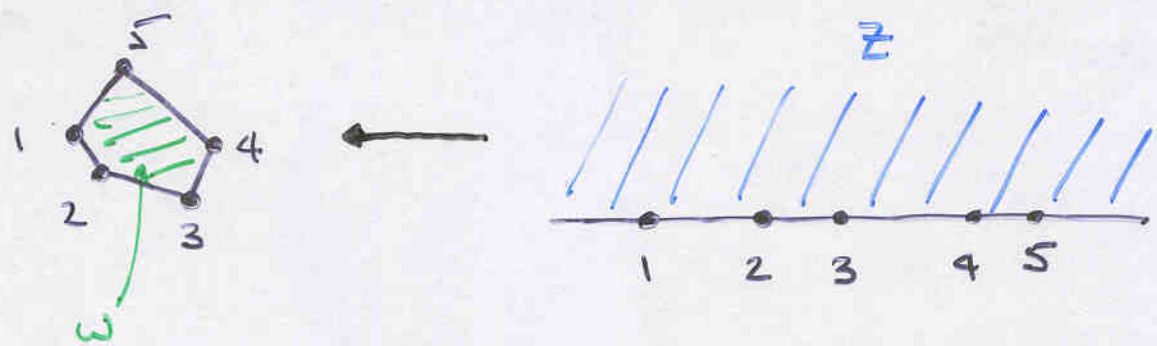
Point	z	w
1	+1	+1
2	0	a
3	-1	-1

$$\frac{z-1}{z} \Big/ \frac{-2}{-1} = \frac{w-1}{w-a} \Big/ \frac{-2}{-1-a} \Rightarrow z = -\frac{1}{a}\,\frac{w-a}{w-1/a}$$

$$\varphi(x,y) = \mathrm{Re}\left[\log(z)\right] = \mathrm{Re}\left[\log\left(-\frac{1}{a}\cdot\frac{w-a}{w-1/a}\right)\right]$$

$$\varphi(x,y) \propto \log\left[\frac{(a^2-1)x + a(x^2+y^2-1)}{(x-a)^2+y^2}\right]$$

Schwarz - Christoffel Transform



$$w(z) = C_1 \int_0^z (z-z_1)^{\gamma_1 - 1} (z-z_2)^{\gamma_2 - 1} \cdots (z-z_n)^{\gamma_n - 1} \, dz + C_2$$

Notes: (1) γ_i is $\dfrac{\text{interior angle of final object}}{\pi}$

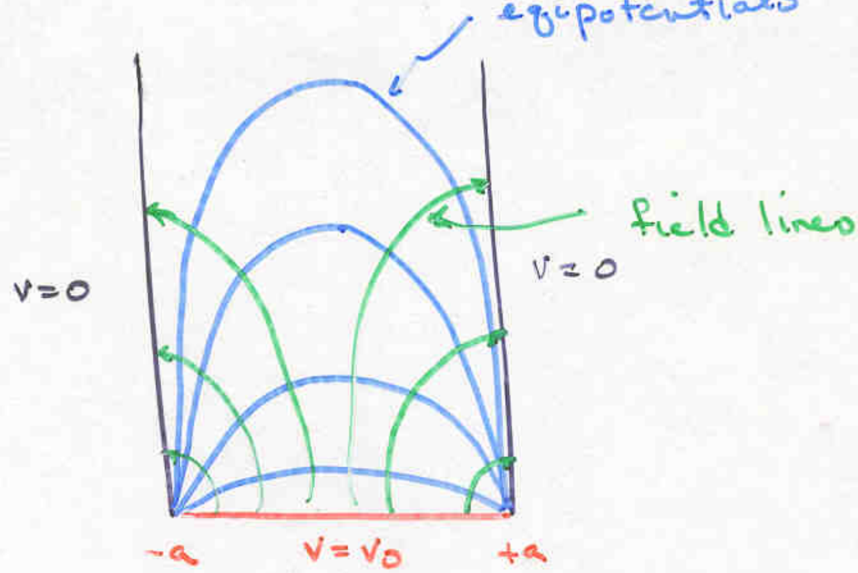
(2) $\sum_{i=1}^{n} \gamma_i = n-1$

(3) C_1 and C_2 are determined by two values $z_i \to w_i$ $z_j \to w_j$

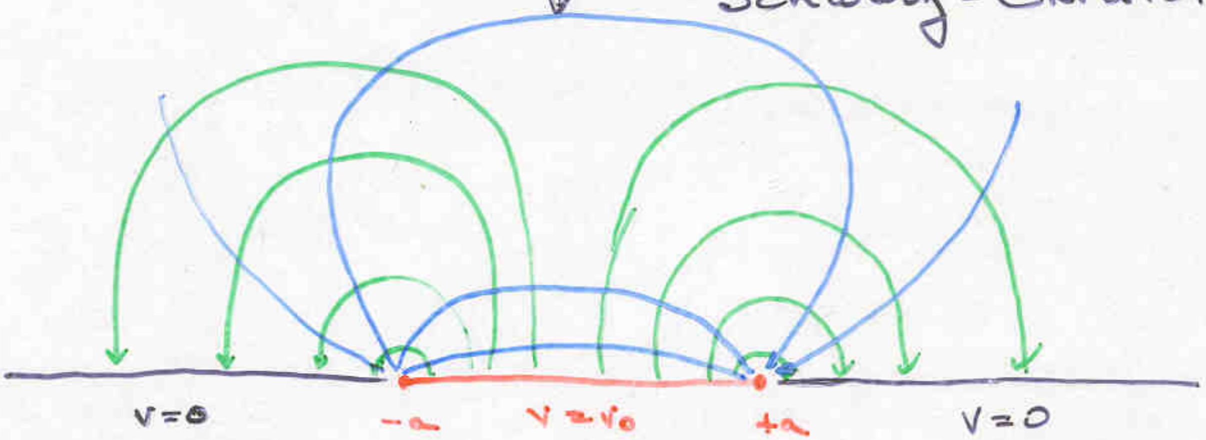
(4) Spacing of other points on real axis is determined by positions of other w-plane points.

(5) Need inverse mapping.

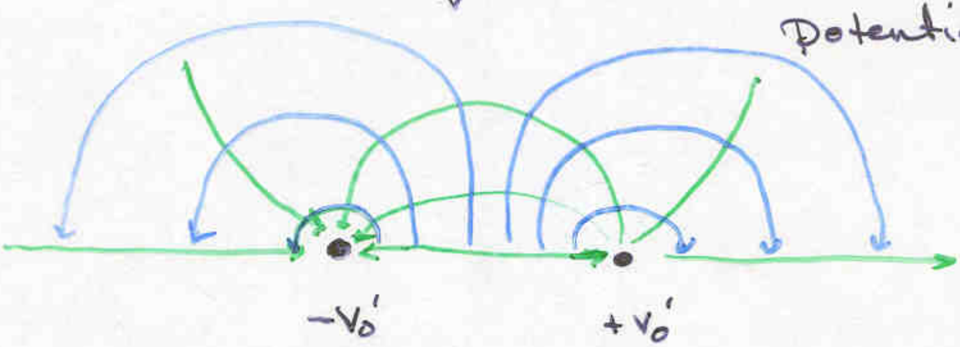
Strip Problem



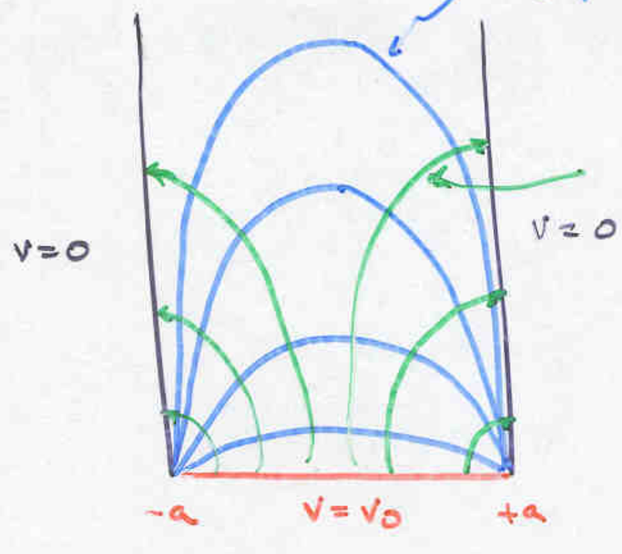
transform — inverse
Schwarz-Christoffel



Invert field and
potential



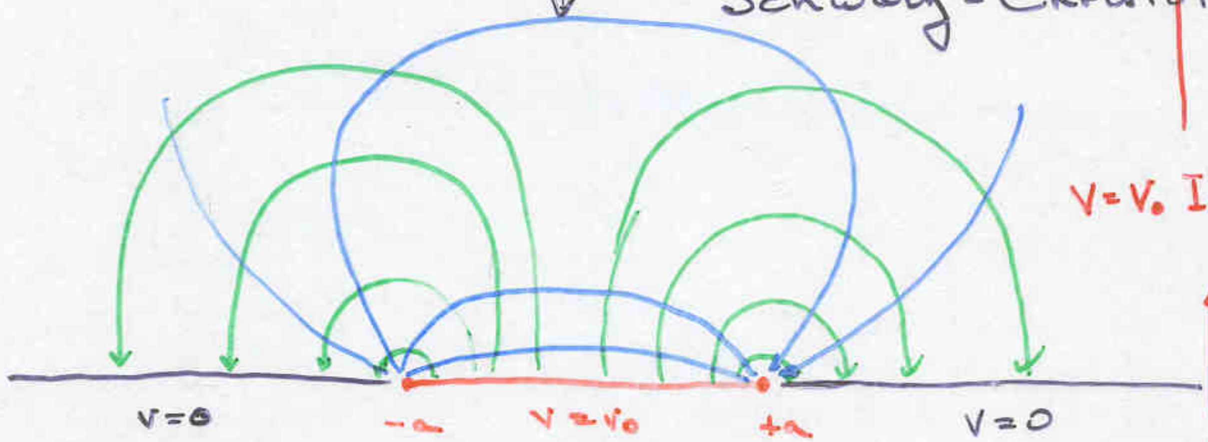
Strip Problem



$$V = V_0 \operatorname{Im} \log \frac{\sin\frac{\pi w}{2a} - 1}{\sin\frac{\pi w}{2a} + 1}$$

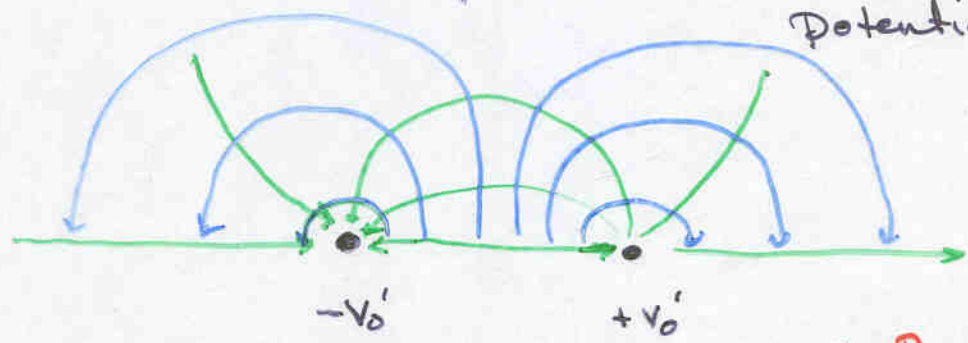
$$z = a \sin \frac{\pi w}{2a}$$

transform — inverse
Schwarz-Christoffel



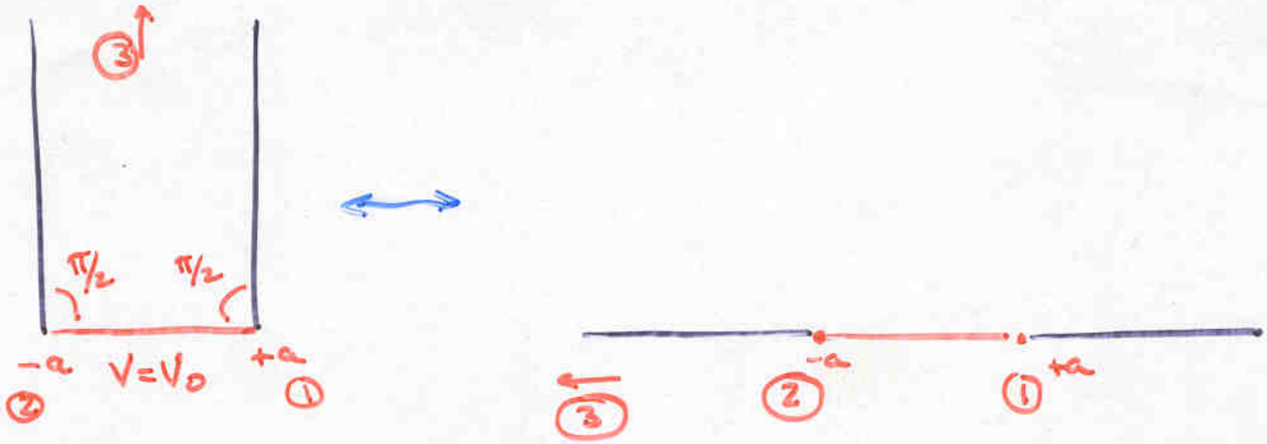
$$V = V_0 \operatorname{Im} \log \frac{(z-a)}{(z+a)}$$

Invert field and potential



$$V = V_0 \operatorname{Re} \log \frac{(z-a)}{(z+a)}$$

Strip Potential - Schwarz Christoffel



point	z_i	ω_i	γ_i
1	a	a	$1/2$
2	$-a$	$-a$	$1/2$
3	∞	∞	0

← points at ∞ don't enter.

$$\omega = C_1 \int_0^z (z-a)^{-1/2} (z+a)^{-1/2} \cancel{(z-\infty)}^{-1} \, dz + C_2$$

$$= C_1 \int_0^z \frac{dz}{\sqrt{z^2-a^2}} + C_2$$

$$= C_1 \operatorname{ArcSin} \frac{z}{a} + C_2$$

$(0 \to 0)$

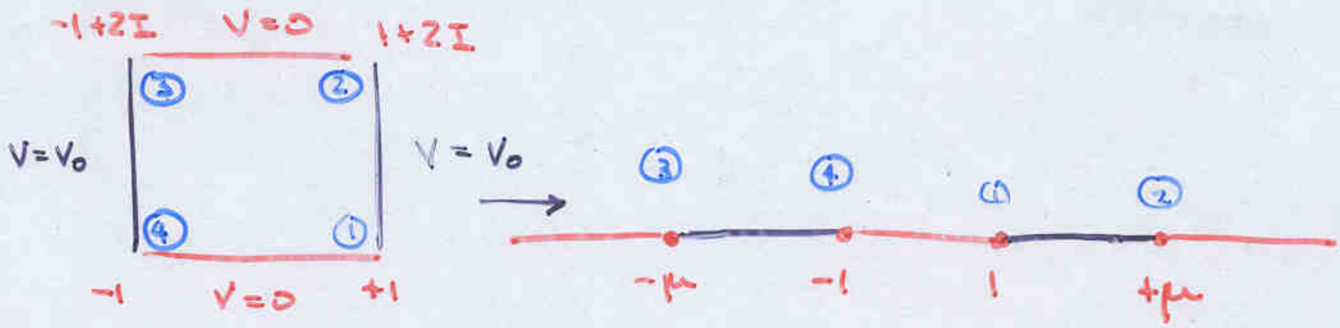
$$C_1 = \frac{2a}{\pi} \qquad C_2 = 0$$

$(a \to a)$ $\qquad (0 \to 0)$

$$\omega = \frac{2a}{\pi} \operatorname{ArcSin}\left(\frac{z}{a}\right) \Rightarrow z = a \operatorname{Sin} \frac{\pi \omega}{2a}$$

$$\varphi = \operatorname{Im}\left[V_0 \log \frac{z-a}{z+a}\right] = \operatorname{Im}\left[V_0 \log \left[\frac{\operatorname{Sin}\frac{\pi\omega}{2a} - 1}{\operatorname{Sin}\frac{\pi\omega}{2a} + 1}\right]\right]$$

Potential in a Square tube



point	z_i	w_i	γ_i
1	1	1	$1/2$
2	μ	$1+2I$	$1/2$
3	$-\mu$	$-1+2i$	$1/2$
4	-1	-1	$1/2$

$$w = C_1 \int_0^z (z-1)^{-1/2} (z-\mu)^{-1/2} (z+\mu)^{-1/2} (z+1)^{-1/2} \, dz + C_2$$

$$= C_1 \int_0^z \frac{dz}{\sqrt{(z^2-1)(z^2-\mu^2)}}$$

$z=0 \Rightarrow w=0 \Rightarrow C_2 = 0$

C_1 and μ determined by other points.

$z=1 \Rightarrow w=1$

$$1 = C_1 \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-z^2/\mu^2)}}$$

Elliptical Integral

$z = \mu \Rightarrow \omega \to 1 + 2i$

$$1 + 2i = c \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-z^2/\mu^2)}} + ic \int_1^{\mu} \frac{dz}{\sqrt{(z^2-1)(1-z^2/\mu^2)}}$$

$$\Rightarrow c = \frac{2}{\int_1^{\mu} \frac{dz}{\sqrt{(z^2-1)(1-z^2/\mu^2)}}}$$

and μ is solution to equation:

$$1 = \frac{2 \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-z^2/\mu^2)}}}{\int_1^{\mu} \frac{dz}{\sqrt{(z^2-1)(1-z^2/\mu^2)}}}$$

← elliptical integral

← elliptical integral after $z = \sqrt{\mu}\left[\frac{1+y\alpha}{1-y\alpha}\right]$

$\alpha = \frac{\sqrt{\mu}-1}{\sqrt{\mu}+1}$

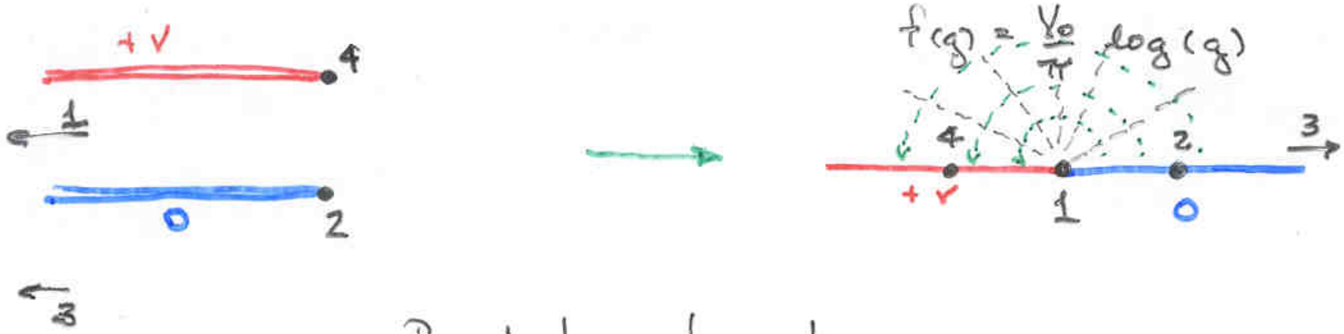
Next get inverse function:

$z = f^{-1}(\omega)$

← Jacobi Sine functions

$z = \mathrm{JacobiSN}\left[\omega/c,\ 1/\mu^2\right]$

EDGE OF A CAPACITOR



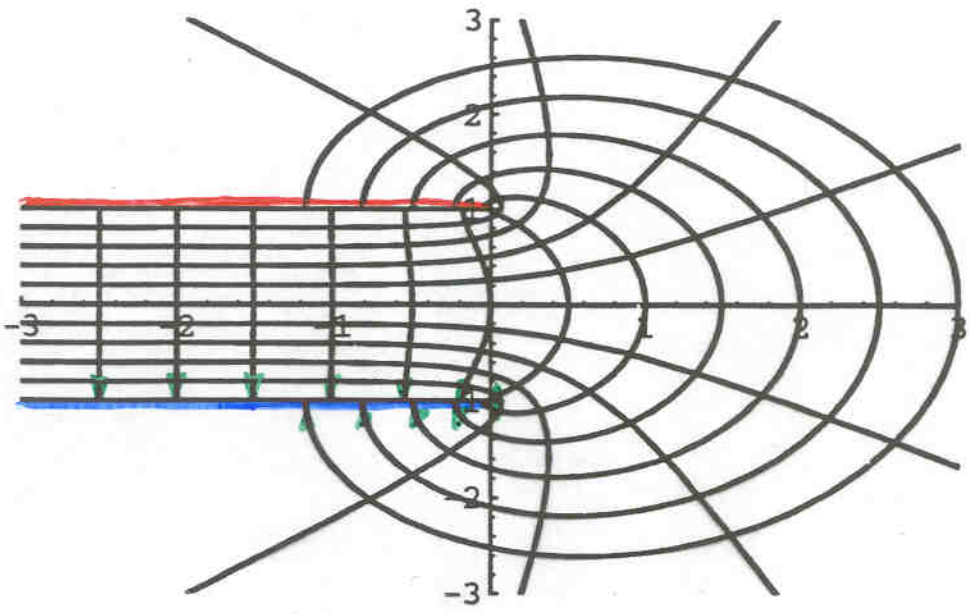
Point	z_i	α_i	g_i
1	$-\infty$	0	0
2	$-ih$	2	+1
3	$-\infty$	-2	∞
4	$+ih$	2	-1

$$z = C_1 \int_{g_0}^{g} \frac{1}{g}(g-1)(g+1)\, dg + C_2$$

$$= C_1 \int_{g_0}^{g} \frac{g^2-1}{g} dg + C_2 = C_1' \left(\frac{g^2}{2} - \log g\right) + C_2$$

$$= \frac{-h}{\pi}\left(g^2 - 2\log g + i\pi - 1\right)$$

(not invertable)



Ellipse in a Constant External field at 45°

Steps

Logic

1) Convert ellipse to circle
2) Rotate circle
3) Find analytic function for circle in uniform field

Solution

Step 3:

$$\varphi(x,y) = y\left(1 - \frac{r_0^2}{x^2+y^2}\right) \Rightarrow f(z) = z + \frac{r_0^2}{z}$$

analytic function $\varphi = \mathrm{Im}(f(z))$

Step 2: Rotation ⇒ linear transform

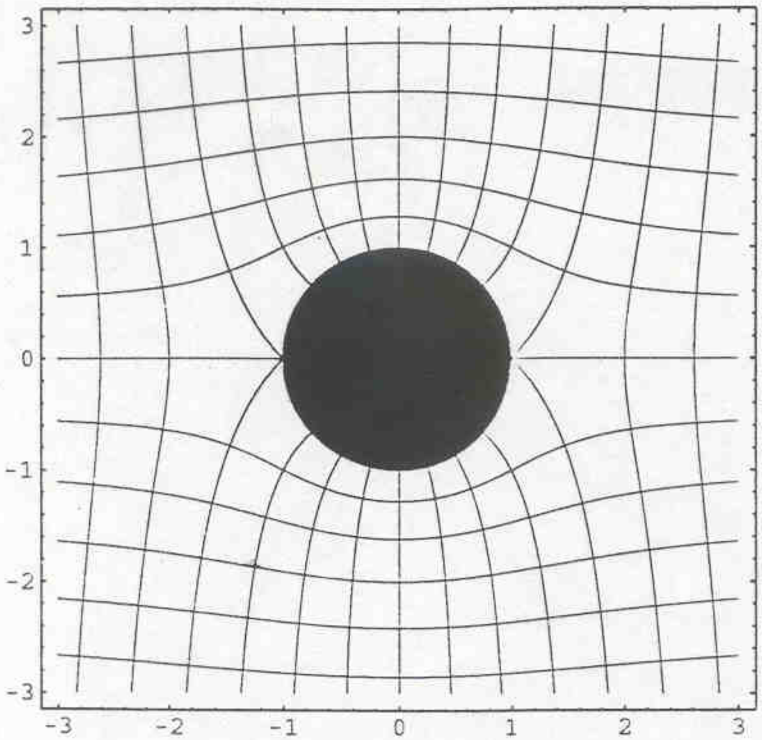
$$z = g\, e^{-i\pi/4}$$

Step 3: Ellipse to circle:

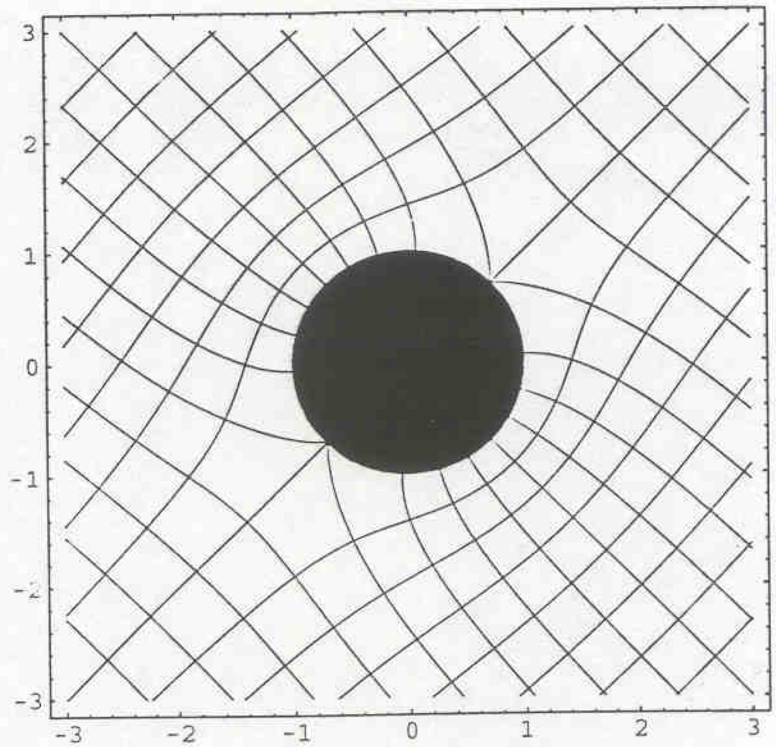
$$g = w + \sqrt{w^2 - r^2}$$

Inverse. $\left[w = \frac{r}{2}\left[\frac{g}{r} + \frac{r}{g}\right] \right]$

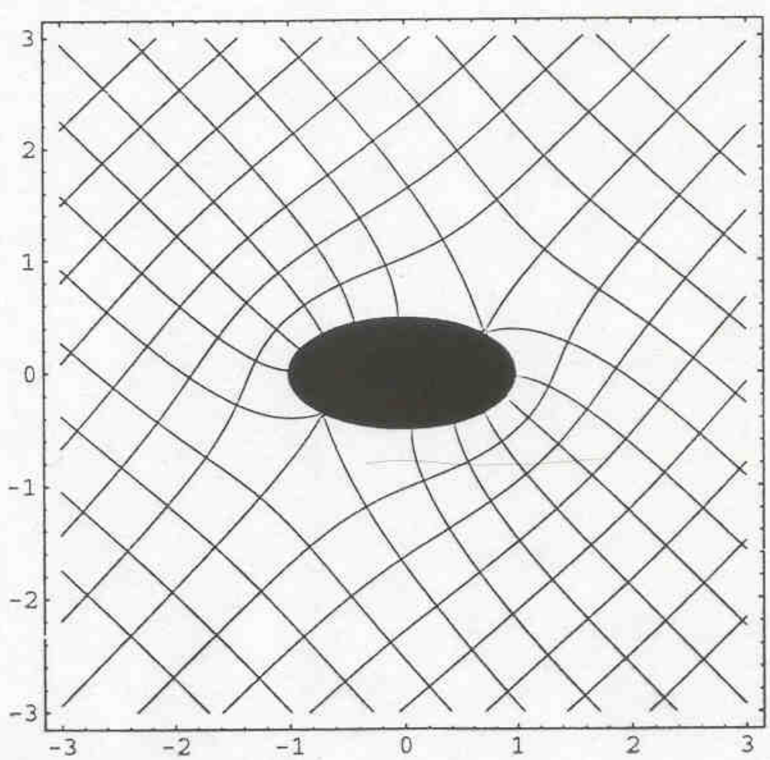
Inverse: Exterior of circle → whole plane.
Radial lines → hyperbola
Exterior Circles → ellipses.



Dipole potential for 2-D: Image charges



Linear transforms: Rotation $z = w e^{i\pi/4}$



Transform which takes circles into ellipses.