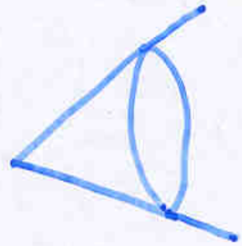
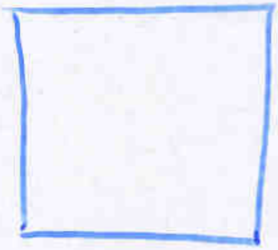
SPHERICAL COORDINATE PROBLEMS

Typical boundary conditions

spheres
constant r



cones
constant θ



planes
constant φ

Laplaces equation:

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\Phi}_{\frac{1}{r}\frac{\partial^2}{\partial r^2} r\Phi} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi = R(r)\,\Theta(\theta)\,Q(\phi) = \frac{U(r)}{r} P(\theta)\, Q(\phi)$$

Jackson

Separation Equation	Solution
$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} R - \frac{\ell(\ell+1)}{r^2} R = 0$	$R = A r^{\ell} + B r^{-\ell-1}$
$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta(\theta) = 0$	[?]
$\frac{\partial^2 Q}{\partial\phi^2} + m^2 Q = 0$	$Q = A e^{+im\phi} + B e^{-im\phi}$ $m \neq 0$; $= A + B\phi$ if $m = 0$

LEGENDRE POLYNOMIALS

$$\frac{d}{dz}(1-z^2)\frac{dP}{dz} + \left[l(l+1) - \frac{m^2}{1-z^2}\right]P = 0$$

(separated spherical equation if $z = \cos\theta$)

First study case where $m = 0$ $\left[\frac{\partial Q}{\partial \phi} = 0\right]$

$$\frac{d}{dz}(1-z^2)\frac{dP(z)}{dz} + l(l+1)P(z) = 0$$

Try: $P(z) = \sum_{i=0}^{\infty} a_i z^i$ $\quad -1 \le z \le 1$

terms regular over region

$$\sum_{i=0}^{\infty} a_i \frac{d}{dz}(1-z^2)\frac{dz^i}{dz} + \sum_{i=0}^{\infty} a_i\, l(l+1)\, z^i = 0$$

$$\sum_{i=\underline{1}}^{\infty} a_i (i)\frac{d}{dz}(1-z^2)\, z^{i-1} + \sum_{i=0}^{\infty} a_i\, l(l+1)\, z^i = 0$$

$$\sum_{i=\underline{2}}^{\infty} a_i (i)(i-1)\, z^{i-2} - \sum_{i=1}^{\infty} a_i (i)(i+1)\, z^i + \sum_{i=0}^{\infty} a_i\, l(l+1) z^i = 0$$

$i' = i-2$ $\quad \sum_{i'=0}^{\infty} a_{i'+2}(i'+2)(i'+1)\, z^{i'} - \sum_{i=0}^{\infty} a_i\left[(i)(i+1) - l(l+1)\right] z^i = 0$

$$\sum_{i=0}^{\infty} z^i \left\{a_{i+2}(i+2)(i+1) - a_i\left[i(i+1) - l(l+1)\right]\right\} = 0$$

LEGENDRE POLYNOMIALS

COEFFICIENT RECURSION RELATION:

$$a_{i+2} = a_i \frac{[i(i+1) - \ell(\ell+1)]}{(i+2)(i+1)}$$

COMMENTS: (1) a_0, a_1 undefined.

(2) sequence terminates when $i = \ell$.

[series will diverge at ± 1 unless it terminates]

$\Rightarrow$ either a_0 or $a_1 = 0$.

; ℓ = integer.

Legendre Polynomials:

$$P_0 = 1 \qquad\qquad P_\ell \equiv 1 \text{ at } z = 1$$

$$P_1 = x$$

$$P_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3 = \frac{5x^3}{2} - \frac{3}{2}x$$

$$P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$$

for P_5 $\quad a_3 = -\frac{14}{3}a_0 \quad a_5 = -\frac{4}{10}a_3 = +\frac{21}{5}a_0$

$$1 = \sum a_i = 1 + \frac{21}{5} - \frac{14}{3} = \frac{8}{15}a_0 \Rightarrow a_0 = \frac{15}{8}$$

$$P_5 = \frac{63}{8}x^5 - \frac{70}{8}x^3 + \frac{15}{8}x$$

Legendre Polynomials: Orthogonality

$$\int_{-1}^{1} P_{\ell'}(z)\, P_\ell(z)\, dz =$$

$$= \int_{-1}^{1} P_{\ell'} \left[\frac{d}{dz}(1-z^2)\frac{d}{dz}P_\ell\right] dz \left[-\frac{1}{\ell(\ell+1)}\right]$$

$$= -\frac{1}{\ell(\ell+1)}\left[\left. P_{\ell'}\cancelto{0}{(1-z^2)}\frac{d}{dz}P_\ell\right|_{-1}^{1} - \int_{-1}^{1}\frac{dP_{\ell'}}{dz}(1-z^2)\frac{dP_\ell}{dz}\,dz\right]$$

$$= \frac{1}{\ell(\ell+1)}\int_{-1}^{1}\frac{dP_\ell}{dz}(1-z^2)\frac{dP_{\ell'}}{dz}\,dz$$

$$= \frac{1}{\ell(\ell+1)}\left[\left. P_\ell\cancelto{0}{(1-z^2)}\frac{dP_{\ell'}}{dz}\right|_{-1}^{1} - \int_{-1}^{1}P_\ell\frac{d}{dz}(1-z^2)\frac{dP_{\ell'}}{dz}\,dz\right]$$

$$= \frac{-1}{\ell(\ell+1)}\int_{-1}^{1}P_\ell\left[-\ell'(\ell'+1)P_{\ell'}\right]dz$$

$$\int_{-1}^{1}P_{\ell'}P_\ell\,dz = \frac{\ell'(\ell'+1)}{\ell(\ell+1)}\int_{-1}^{1}P_\ell P_{\ell'}\,dz$$

$$\left[\ell(\ell+1) - \ell'(\ell'+1)\right]\int_{-1}^{1}P_\ell P_{\ell'}\,dz = 0$$

$$\Rightarrow \int_{-1}^{1}P_\ell P_{\ell'}\,dz = 0 \quad \text{if } \ell \neq \ell'$$

Legendre Polynomials
Properties

(1) Generating function:

$$\Phi(t,z) = \frac{1}{\sqrt{1+t^2-2tz}} = \sum_{n=0}^{\infty} t^n P_n(z)$$

$$\frac{1}{n!}\left.\frac{\partial^n \Phi}{\partial t^n}\right|_{t=0} = P_n(z)$$

(2) Rodrigeus formula

$$P_n = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n$$

(3) Normalization:

$$\int_{-1}^{1} P_\ell^2 \, dz = \frac{2}{2\ell+1}$$

(4) Recursion Relationships:

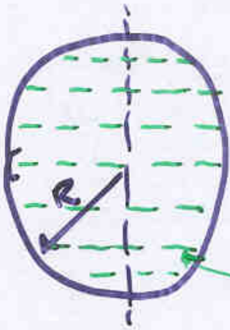
$$\frac{dP_{\ell+1}(z)}{dz} - (2\ell+1)P_\ell(z) - \frac{dP_{\ell-1}(z)}{dz} = 0$$

$$(\ell+1)P_{\ell+1}(z) - z(2\ell+1)P_\ell(z) + \ell P_{\ell-1}(z) = 0$$

$$(1-z^2)\frac{P_\ell}{z} + \ell z P_\ell - \ell P_{\ell-1} = 0$$

$$\frac{dP_{\ell+1}}{dz} - z\frac{dP_\ell}{dz} - (\ell+1)P_\ell = 0$$

Inside a sphere with $V = V_0 \cos\theta$ on surface



$$V(R,\theta,\phi) = V_0 \cos\theta$$

expect these to be equipotentials

$$V(r,\theta) = \sum_{n=0}^{\infty} \left[\left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta) \right]$$

$B_n = 0$ because potential must be finite at $r=0$ [no charge there]

$$\frac{2}{2n+1} A_n R^n = \int_{-1}^{1} V(R,\theta)\, P_n(\cos\theta)\, d\cos\theta$$

$$= \int_{-1}^{1} V_0\, P_1 P_n\, dz = V_0\, \delta_{n1} \cdot \frac{2}{2\cdot 1+1}$$

$$\Rightarrow A_1 = \frac{V_0}{R} \qquad \Rightarrow V(r,\theta,\phi) = \frac{V_0}{R} \cdot \overset{P_1(z)}{\underbrace{r\cos\theta}_{z}}$$

uniform field

How about outside:

Insufficient information

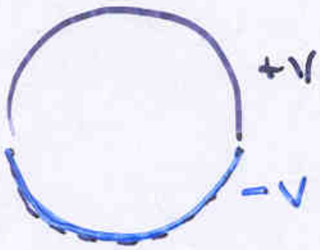
if $V \to 0$ as $r \to \infty$ then $A_n = 0$

$$\frac{2}{3} \frac{B_1}{R^2} = \frac{2}{3} V_0$$

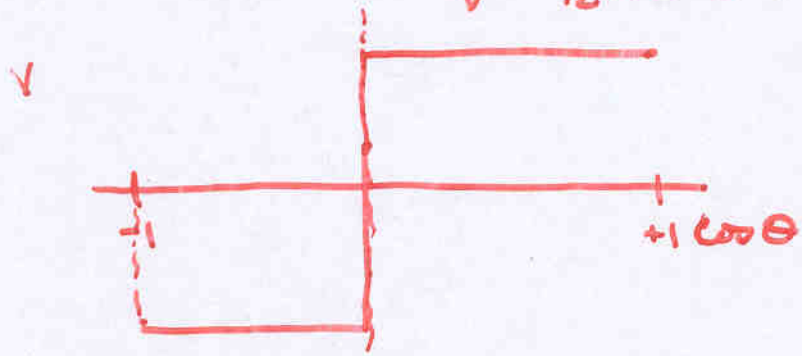
$$V(r,\theta,\phi) = V_0 \frac{R^2}{r^2} P_1(z) = V_0 \frac{R^2}{r^2} \cos\theta$$

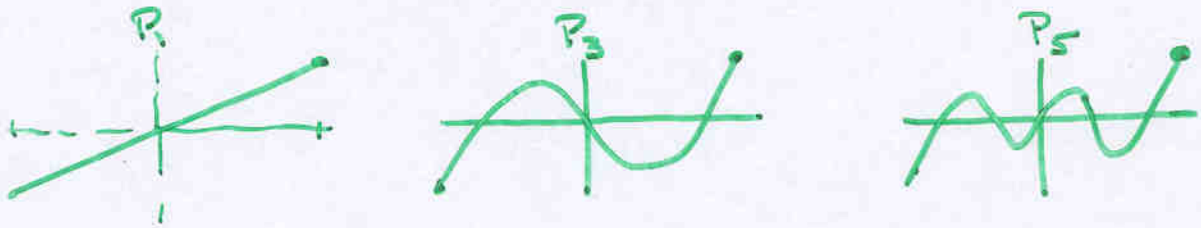
dipole field.

Split potential sphere



$$V = +V_0 \text{ if } 0 \le \theta < \pi/2$$
$$= -V_0 \text{ if } \pi/2 \le \theta \le \pi$$





odd function $\Rightarrow \int V(R,\theta)\, P_n\, dz = 0$ for n = even

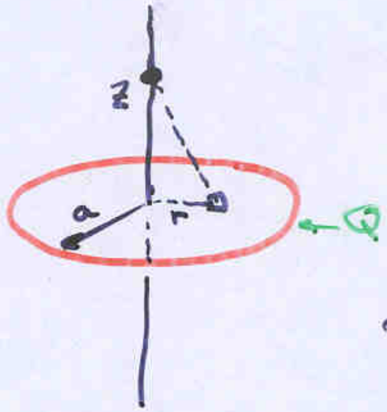
For P_n with n odd.

$$\frac{2}{2n+1} R^n A_n = -\int_{-1}^{0} V_0 P_n\, dz + \int_0^1 V_0 P_n\, dz = 2V_0 \int_0^1 P_n\, dz$$
$$= \frac{2V_0}{2n+1} \int_0^1 \frac{d}{dz}\left[P_{n+1} - P_{n-1}\right] dz$$
$$= \frac{2V_0}{2n+1}\left[P_{n+1} - P_{n-1}\right]_0^1 = \frac{2V_0}{2n+1}\left[P_{n-1}(0) - P_{n+1}(0)\right]$$

if ℓ even
$$P_\ell(0) = (-1)^{\ell/2} \frac{1\cdot 3\cdot 5\cdots(\ell-1)}{2\cdot 4\cdot 6\cdots \ell} = \frac{(-1)^{\ell/2}(\ell-1)!!}{2^{\ell}\cdot(\ell/2)!}$$
$$\{P_\ell(0)\}_{\ell = \text{even}} = \left\{1, -\frac{1}{2}, +\frac{3}{8}, -\frac{5}{16}, \cdots\right\}$$

$$V(r,\theta,\phi) = V_0\left[\frac{3}{2}\frac{r}{R} P_1(z) - \frac{7}{8}\frac{r^3}{R^3} P_3(z) + \frac{11}{16}\frac{r^5}{R^5} P_5(z) + \cdots\right]$$

POTENTIAL FROM A DISK OF CHARGE



$$\sigma = \frac{Q}{\pi a^2}$$

Potential along axis

$$V(z) = \int_0^{2\pi}\int_0^a \frac{\sigma}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+z^2}}\, r\,dr\,d\theta$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_0^a (r^2+z^2)^{-1/2}\, r\,dr$$

$$= \frac{2\pi\sigma}{4\pi\varepsilon_0}\left[\sqrt{z^2+a^2} - z\right] = \frac{Q}{2\pi\varepsilon_0}\frac{1}{a^2}\left[\sqrt{z^2+a^2} - |z|\right]$$

if $z \geq a$, $z > 0$: $\sqrt{z^2+a^2} = |z|\sqrt{1+\frac{a^2}{z^2}} = |z|\left[1 + \frac{1}{2}\frac{a^2}{z^2} - \frac{1}{8}\frac{a^4}{z^4} + \frac{1}{16}\frac{a^6}{z^6} + \cdots\right.$

$$\Rightarrow \quad V(z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{|z|} - \frac{1}{4}\frac{a^2}{|z|^3} + \frac{1}{8}\frac{a^4}{|z|^5} + \cdots\right]$$

$$\Rightarrow \quad V(r,\theta,\phi) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{P_0(\cos\theta)}{r} - \frac{a^2}{4}\frac{P_2(\cos\theta)}{r^3} + \frac{a^4}{8}\frac{P_4(\cos\theta)}{r^5} + \cdots\right]$$

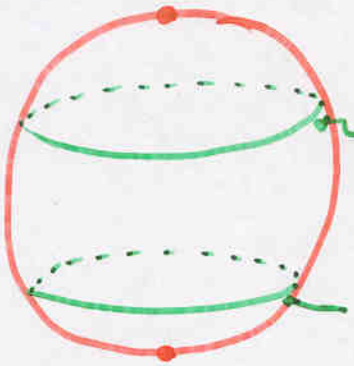
if $z < a \Rightarrow \sqrt{z^2+a^2} = a\sqrt{1+z^2/a^2} \simeq a\left[1 + \frac{1}{2}\frac{z^2}{a^2} - \frac{1}{8}\frac{z^4}{a^4}\right]$

$$V(z) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{a}\left[2 - \frac{2|z|}{a} + \frac{z^2}{a^2} - \frac{1}{4}\frac{z^4}{a^4} + \cdots\right]$$

$$\underset{r<a}{V(r,\theta,\varphi)} = \frac{Q}{4\pi\varepsilon_0 a}\left[2P_0 \mp 2\frac{r}{a}P_1 + \frac{r^2}{a^2}P_2 - \frac{1}{4}\frac{r^4}{a^4}P_4 + \cdots\right]$$

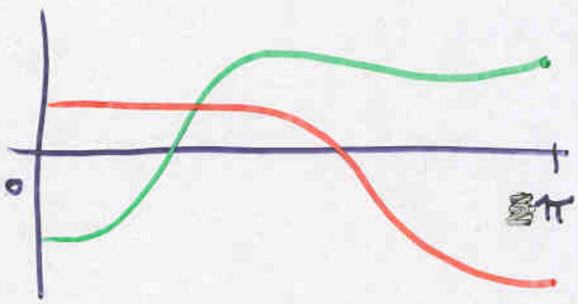
$-$ if $\theta > 0$

Associated Legendre Polynomials and Spherical Harmonics



$$V(\theta_1,\phi) = \left[\sum_{m=1}^{\infty} A^m(\theta_1)\,e^{im\phi} + B^m(\theta_1)\,e^{-im\phi}\right] + V_0(\theta_1)$$

$$V(\theta_2,\phi) = \left[\sum_{m=1}^{\infty} A^m(\theta_2)\,e^{im\phi} + B^m(\theta_2)\,e^{-im\phi}\right] + V_0(\theta_2)$$



⇒ Describe A^m & B^m in a complete set of polynomials for each "m"

$$A^m(\theta) = \sum_{\ell=m,-m}^{\infty,-\infty} A_{\ell m}\,\underbrace{P_\ell^m(\cos\theta)}_{\text{associated Legendre polynomials}}$$

$$\frac{d}{dz}(1-z^2)\frac{dP_\ell^m}{dz} + \left\{\ell(\ell+1) - \frac{m^2}{1-z^2}\right\}P_\ell^m(z) = 0$$

$$P_\ell^0(z) = P_\ell(z) = \text{Legendre polynomials}$$

$m \geq 0$ $$P_\ell^m(z) = (-1)^m (1-z^2)^{m/2}\frac{d^m}{dz^m}P_\ell(z) = \frac{(-1)^m}{2^\ell\,\ell!}(1-z^2)^{\frac{m}{2}}\frac{d^{m+\ell}}{dz^{m+\ell}}(z^2-1)^\ell$$

$-m<0$ $$P_\ell^{-m}(z) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(z)$$

ASSOCIATED LEGENDRE POLYNOMIALS

Solution to: $\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}P_\ell^m + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_\ell^m(\cos\theta) = 0$

RECURSION RELATIONSHIPS

$$(2\ell+1)\,u\,P_\ell^m(u) = (\ell-m+1)\,P_{\ell+1}^m(u) + (\ell+m)\,P_{\ell-1}^m(u)$$

$$P_\ell^{m+1} + \frac{2mu}{\sqrt{1-u^2}}\,P_\ell^m(u) + (\ell-m+1)(\ell+m)\,P_\ell^{m-1} = 0$$

Normalization:

$$\int P_{\ell'}^m(x)\,P_\ell^m(x)\,dx = \frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell\ell'}$$

For each m; $\ell \geq m$ gives orthogonal series.

$$\Rightarrow A(x) = \sum_{\ell=m}^{\infty} A_\ell\,P_\ell^m(x)$$

Each is complete

$\Rightarrow$ spherical harmonics are complete on sphere.

Properties of Spherical Harmonics

$$Y_{\ell m}(\theta, \varphi) = \Theta_{\ell m}(\theta)\,\Phi_m(\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\phi}$$

symmetry: $$Y_{\ell,-m}(\theta,\phi) = (-1)^m\, Y^*_{\ell,m}(\theta,\varphi)$$

orthogonality: $$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

completeness: $$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell,m}(\theta,\phi) = \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')$$

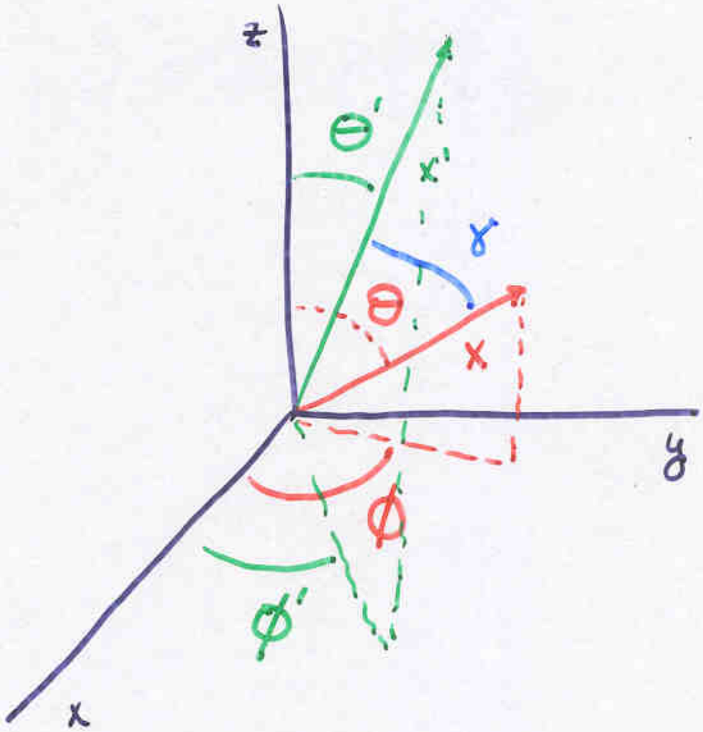
symmetry: $$Y_{\ell,m}(\pi-\theta, \pi+\varphi) = (-1)^\ell\, Y_{\ell,m}(\theta,\varphi)$$

Expansion:

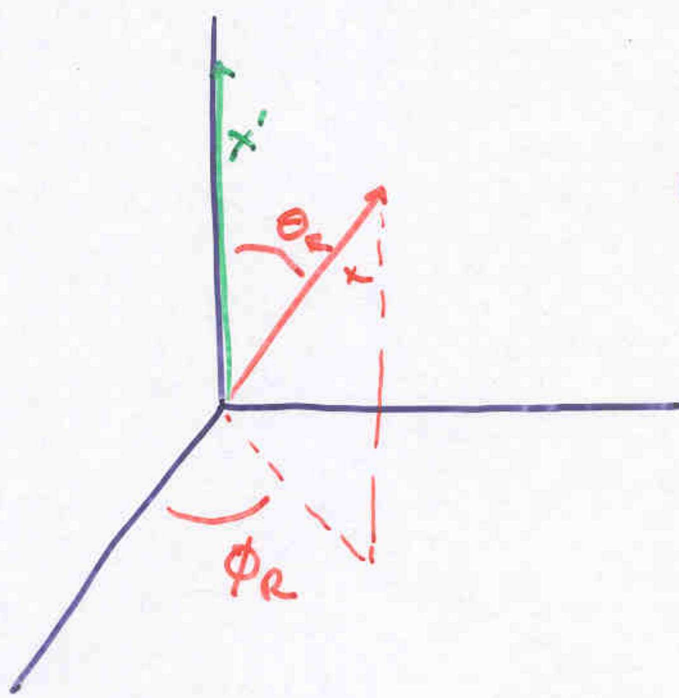
$$g(\theta,\phi) = \sum_{\ell=0,\, m=-\ell}^{\infty,\ \ell} A_{\ell m}\, Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow\; A_{\ell m} = \int_0^{2\pi}\int_{-1}^{1} d\phi\, d\cos\theta\;\, g(\theta,\phi)\, Y^*_{\ell,m}(\theta,\phi)$$

Regular System



Rotated System



$\theta_R = \gamma$

Addition Theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

Expand $P_\ell(\cos\gamma)$ in $Y_{\ell m}(\theta,\phi)$

$$P_\ell(\cos\gamma) = \sum_{\ell'=0}^{\infty}\sum_{m=-\ell'}^{\ell'} A_{\ell' m}(\theta'\phi')\,Y_{\ell' m}(\theta,\phi)$$

Because $\quad \nabla^2_\gamma P_\ell = -\frac{\ell(\ell+1)}{r^2} P_\ell$

$$\nabla^2_{\theta,\phi} Y_{\ell m} = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}$$

$$\nabla^2_\gamma = \nabla^2_{\theta\phi} \Rightarrow \ell' = \ell$$

$$\Rightarrow P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} A_{\ell m}(\theta'\phi')\,Y_{\ell m}(\theta,\phi)$$

$$A_{\ell m} = \int Y^*_{\ell m} P_\ell(\cos\gamma)\, d\Omega$$

Expand $Y^*(\theta,\phi)$ in terms of rotated θ_R, ϕ_R

$$\sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}(\theta,\phi) = \sum_{m'=-\ell}^{\ell} B_{\ell m'}(m)\, Y_{\ell m'}(\theta_R,\phi_R)$$

$$A_{\ell m} = B_{\ell 0}(m)$$

Addition Theorem
(cont.)

Special case: when $(\theta, \phi) = (\theta', \phi')$

$\Rightarrow \theta_R = 0$

$$Y^*_{\ell m}(\theta', \phi') = Y^*_{\ell m}(\theta, \phi)\Big|_{\theta_R=0} = \sqrt{\frac{2\ell+1}{4\pi}} \sum B_{\ell m'}(m)\, Y_{\ell m'}(\theta_R \phi_R)\Big|_{\theta_R=0}$$

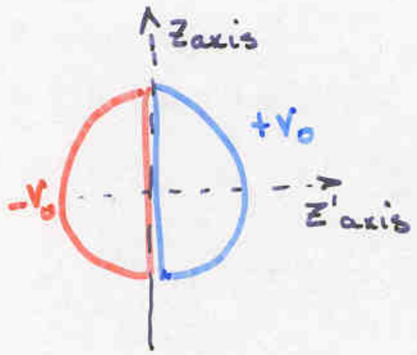
$$Y_{\ell m}(0, \phi) = 0 \quad \text{if } m \neq 0$$

$$= \sqrt{\frac{2\ell+1}{4\pi}} \quad \text{if } m = 0$$

$\Rightarrow$

$$A_{\ell m} = B_{\ell 0}(m) = \frac{4\pi}{2\ell+1} Y^*_{\ell m}(\theta', \phi')$$

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi)$$



Methods of solving:

(1) Substitution of variables in final answer

$$\varphi(r'(r,\theta,\phi),\ \theta'(r,\theta,\phi),\ \phi'(r,\theta,\phi)) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta')$$

(2) Expansion in spherical harmonics:

$$\varphi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\varphi)$$

$$A_{lm}^{(R)} = \int \varphi(R,\theta,\phi)\, Y_{lm}^{*}\, d\Omega$$

(3) Use Addition theorem for spherical harmonics

$$\varphi(r',\theta',\varphi') = \sum_{l=0}^{\infty} \underline{\underline{A_{l}(r')}}\, P_l(\cos\theta')\, r^{l}$$

$$P_l(\cos\theta') = \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}^{*}(\theta,\phi)\, Y_{lm}(\theta_{axis}, \varphi_{axis})$$

from cylindrically symmetric problem:

$$A_l(r') = \frac{V_0}{R^l}\left[P_{l-1}(0) - P_{l+1}(0)\right]$$

Green Function Expansion
Spherical Coordinates

$$G(x,x') = \frac{1}{|x-x'|} + F(x,x')$$

$$\nabla^2 G = -4\pi\,\delta^3(x-x') \qquad \nabla^2 F = 0$$

$\&$ $G(x,x') = 0$ on surface.

$$\frac{1}{|x-x'|} = \sum \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)$$

generating function for P_ℓ

addition theorem

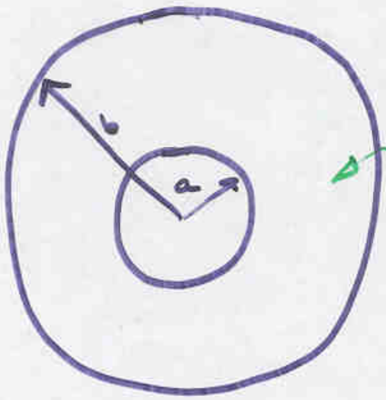
$$F(x,x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m}(\theta,\phi)$$

where $A_{\ell m}$, $B_{\ell m}$ contain θ', ϕ' information

$$G(x,x') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') + A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m}(\theta,\phi)$$

where $A_{\ell m}$ $\&$ $B_{\ell m}$ are chosen such that $G(x,x') = 0$ on boundary.

Example:



find Green's function such that $G(x,a)=0$ and $G(x,b)=0$

Because $Y_{\ell m}$ are orthogonal; each coefficient must $=0$ at each edge.

$$\Rightarrow \frac{4\pi}{2\ell+1}\frac{a^\ell}{r'^{\,\ell+1}}Y^*_{\ell m}(\theta',\phi') + A_{\ell m}a^\ell + \frac{B_{\ell m}}{a^{\ell+1}} = 0$$

$$\frac{4\pi}{2\ell+1}\frac{r'^{\,\ell}}{b^{\ell+1}}Y^*_{\ell m}(\theta',\phi') + A_{\ell m}b^\ell + \frac{B_{\ell m}}{b^{\ell+1}} = 0$$

$$\Rightarrow B_{\ell m} = \frac{\left[1-\left(\frac{b}{r'}\right)^{2\ell+1}\right](r')^\ell\, Y^*_{\ell m}(\theta',\phi')}{\left[1-\left(\frac{b}{a}\right)^{2\ell+1}\right]\frac{2\ell+1}{4\pi}}$$

$$A_{\ell m} = \ldots\ldots$$

$$G(x,x') = \sum_{\ell,m}\frac{4\pi/(2\ell+1)}{1-\left(\frac{a}{b}\right)^{2\ell+1}}Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)\left[r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right]\left[\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right]$$

Limits

$b \to \infty$

$$G(\vec{x},\vec{x}') = \sum_{l,m} \underbrace{\frac{4\pi}{2l+1} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)}_{\text{addition formula}} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{r_<^{l+1} r_>^{l+1}}\right]$$

$$= \underbrace{\sum_l P_l(\cos\gamma) \left[\frac{r_<^l}{r_>^{l+1}}\right.}_{\text{Legendre Polynomial generating function}} \left.- \frac{a^{2l+1}}{r_<^{l+1} r_>^{l+1}}\right]$$

$$= \frac{1}{|x-x'|} - \sum_l P_l(\cos\theta) \frac{\left(a^2/r\right)^l \frac{a}{r}}{(r')^{l+1}}$$

ditto

$$= \frac{1}{|x-x'|} - \underbrace{\frac{a/r}{|a^2/x - x'|}}_{\text{image charge result!}}$$

$a \to 0$

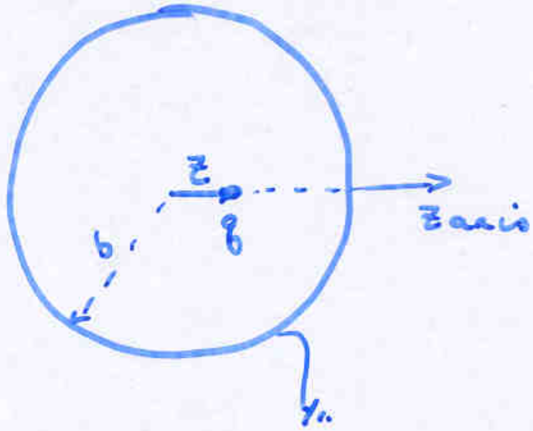
$$G(x,x') = \sum \frac{4\pi}{2l+1} Y^*(\theta',\varphi') Y(\theta,\varphi) \left[\frac{r_<^l}{r_>^{l+1}} - \frac{r_>^l r_<^l}{b^{2l+1}}\right]$$

$$\downarrow$$

$$= \frac{1}{|x-x'|} - \frac{b/|x|}{|b^2/x - x'|}$$

again image result!

Example: Charge inside grounded sphere.



Find surface charge density

$$\sigma = \varepsilon_0 E_\perp = -\varepsilon_0 \frac{\partial \phi}{\partial n} = -\varepsilon_0 \frac{\partial \phi}{\partial r}$$

$$\phi = \int dv' \frac{\rho}{4\pi\varepsilon_0} G(x,x') + \underbrace{\int \frac{\partial G}{\partial n} \phi_s \, ds}_{=0}$$

information about surface charge is in Green's function

$$\rho = q\, \delta(r'-z) \frac{\delta(\cos\theta'-1)}{r'} \frac{\delta(\phi)}{r'}$$

$$Y^*_{lm}(\theta',\phi') \to P_l(1)\sqrt{\frac{4\pi}{2l+1}} = \sqrt{\frac{4\pi}{2l+1}}$$

$Y_{lm} = 0$ for $m \neq 0$

$$G = \sum_{l=0}^{\infty} \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}\, z^l \left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right] \quad \text{for } r > z$$

where evaluated with δ function.

note: $G = 0$ if $r = b$.

$$\phi = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} P_l(\cos\theta)\, z^l \left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right] \quad \text{for } r > z$$

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial r} = \frac{-q}{4\pi b^2} \sum_{l=0}^{\infty} P_l \frac{z^l}{b^l} \cdot [2l+1]$$