Two Dimensional Boundaries
— Polar Coordinates —

$$\frac{\partial \Phi}{\partial z} = 0 \text{ everywhere.}$$

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

$$\Phi(r,\theta) = R(r)\,\Theta(\theta)$$

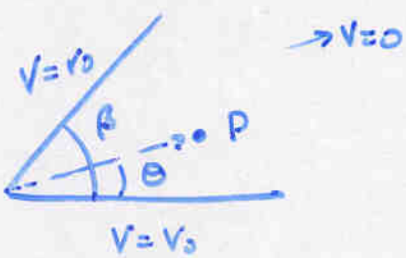
$$r \frac{\partial}{\partial r} r \frac{\partial R}{\partial r} = \nu^2 R \quad \& \quad \frac{\partial^2 \Theta}{\partial \theta^2} = -\nu^2 \Theta$$

$$R_\nu = A r^\nu + B r^{-\nu} \qquad \Theta = C \cos \nu\theta + D \sin \nu \theta$$

if $\nu = 0$ $\quad R_0 = A_0 + B_0 \ln r \qquad \Theta_0 = C_0 + D_0 \theta$

$$\Phi = \sum_{n=0}^{\infty} R_n \Theta_n$$ subject to boundary conditions.

Example:



Boundary conditions imply:

$A_0 = V_0 \qquad C_i = 0$ for $i \geq 1$

$B_0 = 0 \qquad D_0 = 0$

take $\nu = \frac{m\pi}{\beta}$ such that $\sin \nu\theta = 0$ at β

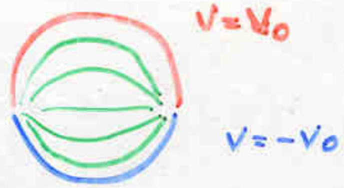
$$\Phi(r,\theta) = V_0 + \sum_{m=1}^{\infty} a_m r^{m\pi/\beta} \sin\left(\frac{m\pi}{\beta}\theta\right)$$

$$\Phi(r,\theta) \simeq V_0 + a_1 r^{\pi/\beta} \sin\left(\frac{\pi\theta}{\beta}\right)$$

$$E_r(r,\theta) = -\frac{\partial \Phi}{\partial r} \simeq -\frac{\pi a_1}{\beta} r^{\frac{\pi}{\beta}-1} \sin\frac{\pi\theta}{\beta} \qquad \left[E_r \to 0 \text{ as } \theta \to 0, \to \beta\right]$$

$$E_\theta(r,\theta) = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{\pi a_1}{\beta} r^{\pi/\beta - 1} \cos\frac{\pi\theta}{\beta}$$

Split Cylinder:



$$\varphi = \left[A_0 + B_0 \log r\right]\left[C_0 + D_0\theta\right] + \sum_{\nu}\left[A_\nu r^\nu + B_\nu r^{-\nu}\right]\left[C_\nu \sin\nu\theta + D_\nu \cos\nu\theta\right]$$

Considerations:

1) φ finite at $r = 0 \Rightarrow B_i = 0$
2) φ is odd function of $\theta \Rightarrow A_0,\ D_i = 0$
3) $\varphi(0) = \varphi(2\pi) \Rightarrow D_0 = 0,\ \nu = n$

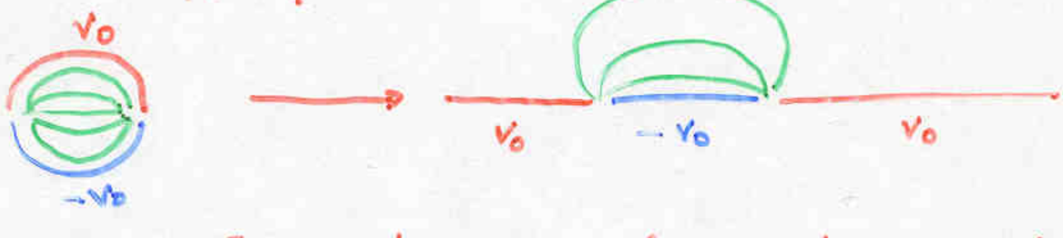
$$\Rightarrow \varphi = \sum A_n r^n \sin n\theta .$$

Coefficients $= A_n = \dfrac{4V_0}{n\pi R^n}$ if n is odd

$= 0$ if n is even

$$\varphi = \sum \frac{4V_0}{n\pi}\left(\frac{r}{R}\right)^n \sin n\theta$$

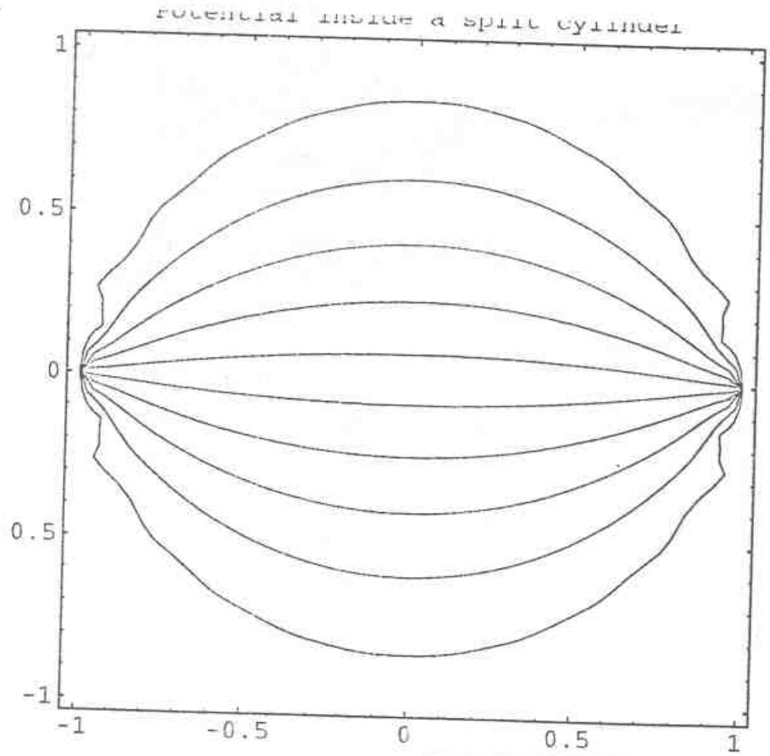
Conformal mapping:



$$z = \frac{w + i}{1 + iw} \qquad f(z) = iV_0 - \frac{2V_0}{\pi}\,\mathrm{Log}\,\frac{z-1}{z+1}$$

$$= \frac{2V_0}{\pi}\log i\left(\frac{z+1}{z-1}\right)$$

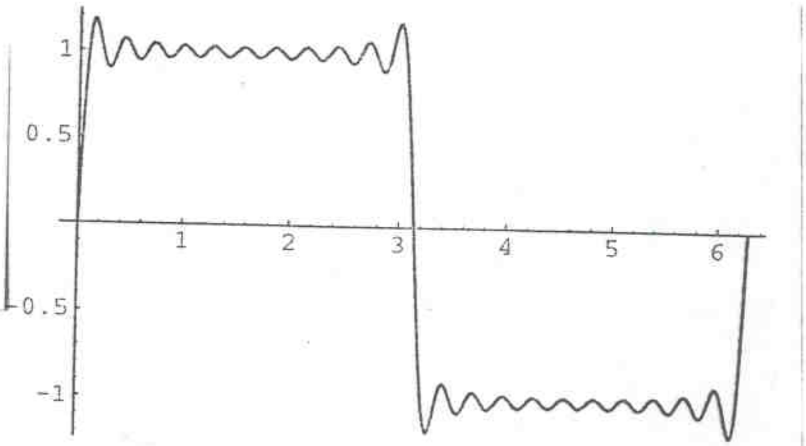
$$\varphi = \mathrm{Im}\,\frac{2V_0}{\pi}\log\frac{1+w}{1-w}$$

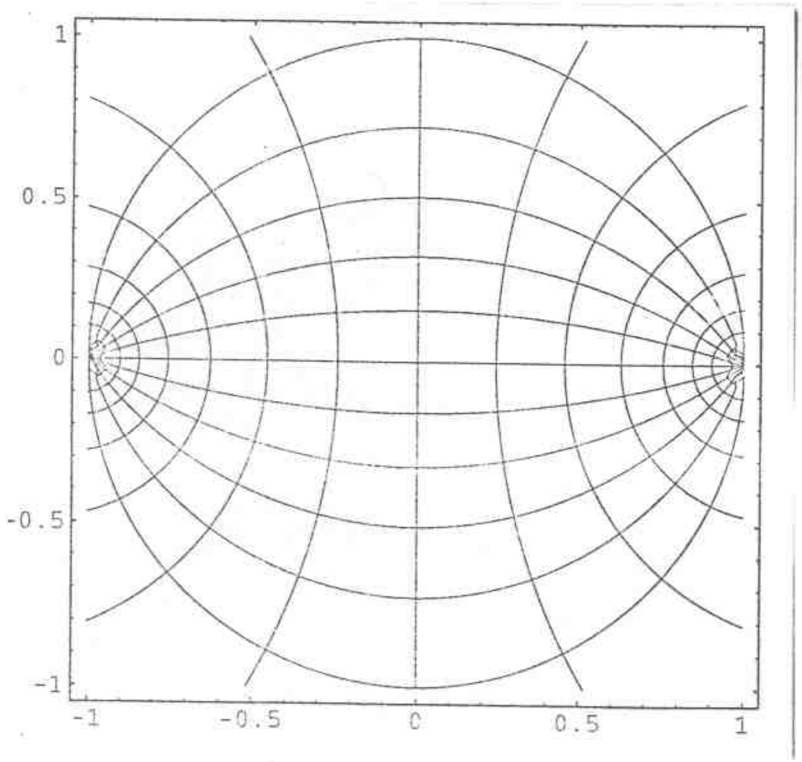


Split Cylinder

Eigenfunction Expansion

$$\varphi = \sum_{\substack{n=1,\\ n=\text{odd}}}^{19} \frac{4V_0}{n\pi}\, r^n \sin n\theta$$

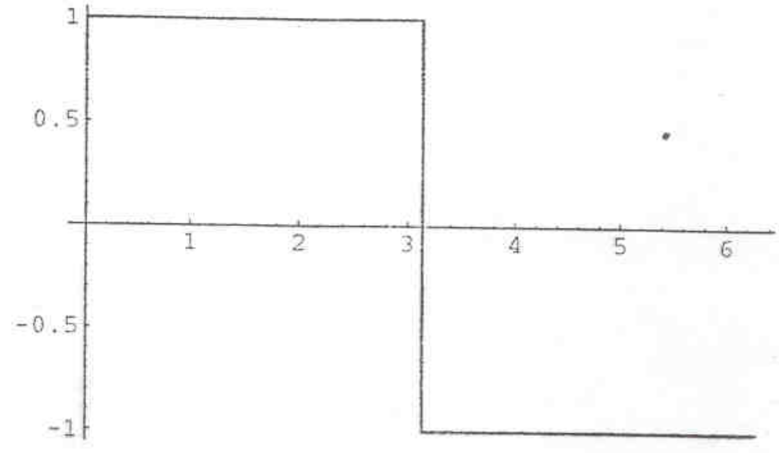




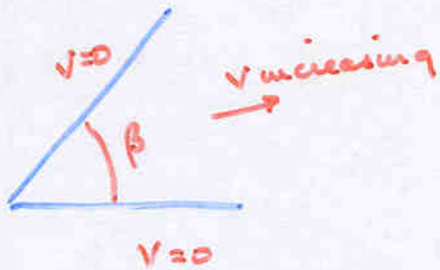
Complex Analysis

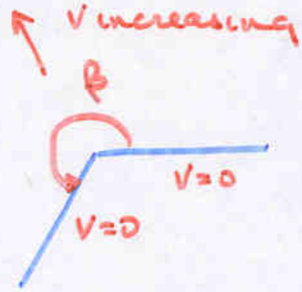
Formal Solution

$$\varphi = \frac{2V_0}{\pi}\, \mathrm{Im} \log \frac{1+z}{1-z}$$



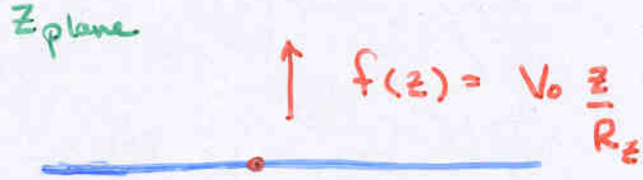
Problem: Find potential in corner or along edge:





Method (1)

Conformal Mapping



$$f(z) = V_0 \frac{z}{R_z}$$

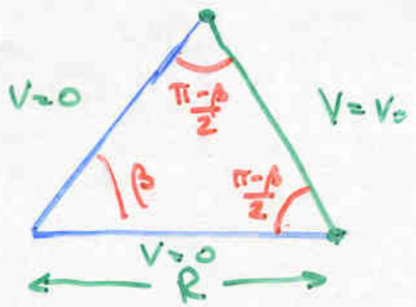
$$\varphi = \text{Im}\, f(z) \qquad z(\omega) = \omega^{\pi/\beta}$$

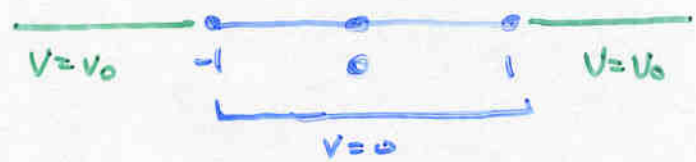
$$\varphi = \frac{V_0}{R_z} \text{Im}\, \omega^{\pi/\beta} = \frac{V_0}{R_z} r^{\pi/\beta} \sin \pi/\beta \cdot \theta$$

$$= V_0 \left(\frac{r}{R_\omega}\right)^{\pi/\beta} \sin \pi/\beta \cdot \theta$$

where R_ω = characteristic distance in "ω" plane.

Method 2 - Schwarz-Christoffel transform:





$$f(z) = \text{"potential" function} = +\frac{V_0}{\pi}\log\left[\frac{1+z}{1-z}\right]$$

Schwarz-Christoffel transform:

$$w = C_1 \int_0^z \frac{1}{z^{1-\beta/\pi}} \frac{1}{(z^2-1)^{\frac{1}{2}(\frac{\beta}{\pi}+1)}} dz + C_2$$

$w = 0$ when $z = 0 \Rightarrow C_2 = 0$

$w = R$ when $z = 1 \Rightarrow C_1 = \dfrac{R}{\int_0^1 \frac{1}{z^{1-\beta/\pi}} \frac{1}{(z^2-1)^{\frac{1}{2}(\beta/\pi+1)}} dz} = \dfrac{R}{C_2}$

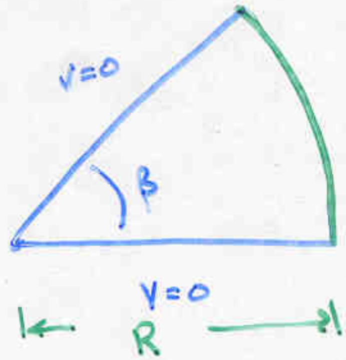
for small z $\quad w \simeq \frac{R}{C_2} \cdot z^{\beta/\pi}$

$$f(z) \simeq \frac{2V_0}{\pi} z \simeq \frac{2V_0}{\pi}\left[C_2 \frac{w}{R}\right]^{\pi/\beta}$$

$$\mathrm{Im}(f) = \frac{2V_0}{\pi} \mathrm{Im}\left(C_2^{\pi/\beta}\right)\left(\frac{r}{R}\right)^{\pi/\beta} \sin\frac{\pi}{\beta}\theta$$

* only good for $\beta < \pi$

Method 3 – Separation of Variables:



Boundary condition separate in R, θ.

$$\varphi = (A_0 + B_0 \log r)(C_0 + D_0 \theta) + \sum_{\nu=1}^{\infty} (A_\nu r^\nu + B_\nu r^{-\nu}) *$$
$$(C_\nu \sin \nu\theta + D_\nu \cos \nu\theta)$$

Boundary Conditions:

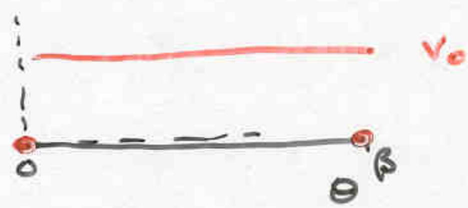
(1) φ finite at $r=0$ $\Rightarrow$ all $B_i = 0$

(2) $\varphi = 0$ at $\theta = 0$ $\Rightarrow$ $A_0 = 0$

(3) $\varphi = 0$ at $\theta = \beta$ $\Rightarrow$ all $D_i = 0$ & $\nu = \frac{n\pi}{\beta}$

$$\varphi = \sum A_n r^{\frac{n\pi}{\beta}} \sin \frac{n\pi}{\beta}\theta$$

at $r = R$ $\quad \varphi =$



$$A_n = \frac{4V_0}{n\pi} \frac{1}{R^{n\pi/\beta}}$$

$$\Rightarrow \varphi = \sum \frac{4V_0}{n\pi} \left(\frac{r}{R}\right)^{n\pi/\beta} \sin \frac{n\pi}{\beta}\theta$$

for small r $\quad \varphi \simeq \frac{4V_0}{\pi}\left(\frac{r}{R}\right)^{\pi/\beta} \sin \frac{\pi}{\beta}\theta$