Separation of Variables – 2 Dimensions

Assume $\frac{\partial \phi}{\partial z} = 0$ everywhere:

$$\Rightarrow \nabla^2 \phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Try: $\phi(x,y) = X(x)\, Y(y)$

$$\Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \alpha^2$$

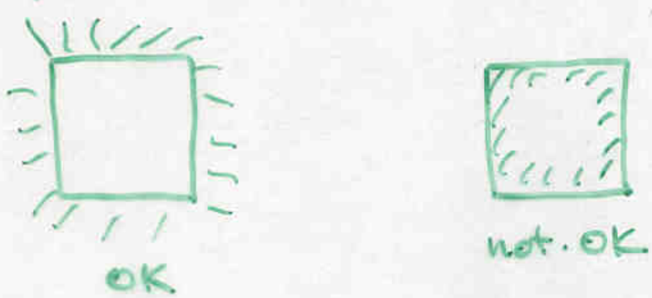
constant

$$\Rightarrow Y(y) = A \sin \alpha y + B \cos \alpha y \qquad \alpha \neq 0$$
$$= Ax + B \qquad \alpha = 0$$
$$X(x) = C \sinh \alpha x + D \cosh \alpha x \qquad \alpha \neq 0$$
$$= Cx + D \qquad \alpha = 0$$

Note: α may be imaginary
$\Rightarrow$ sinh; cosh $\rightarrow$ sin, cos & v.v.

Useful technique if boundary conditions separate in x, y.

Example:



Separation of Variables - 2 Dimensions.

Assume $\frac{\partial \phi}{\partial z} = 0$ everywhere

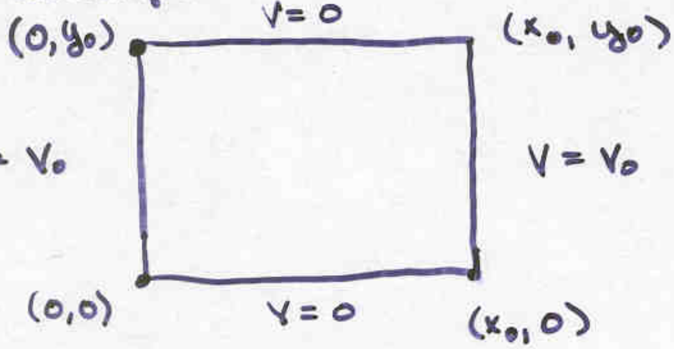
$$\Rightarrow \nabla^2 \phi = 0 \Rightarrow \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

if boundary conditions separate in x, y then:

$$\phi(x,y) = e^{\pm i\alpha x} e^{\pm \alpha y}$$ [α may be imaginary]

$$= (A \sin \alpha x + B \cos \alpha x)(C \sinh \alpha y + D \cosh \alpha y)$$

Example:



Incorrect method:

$A = 0$, $B = V_0$

$\alpha = \frac{2n\pi}{x_0}$

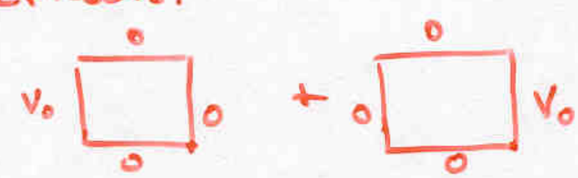
$D = 0$

& Fourier analyze.

$\left.\frac{\partial \phi}{\partial x}\right|_{x=0,\, x_0} = 0$ & really $\frac{\partial \phi}{\partial x} = \infty$ at corner.

Correct methods:

1) solve:



; let $\alpha = \frac{2i n\pi}{y_0}$

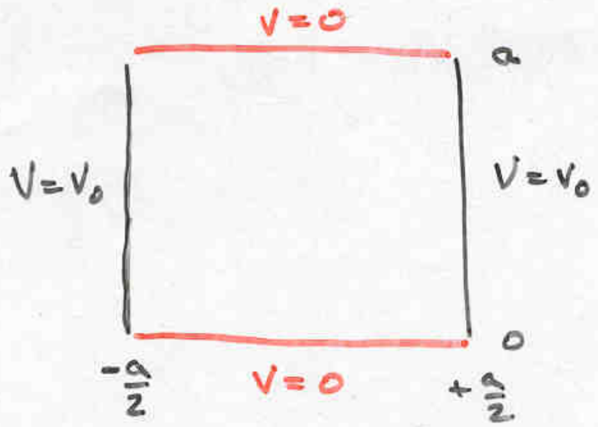
same as before.

2) Again let α be imaginary. solve problem with change of coordinates

$$x' = x - \frac{x_0}{2} \Rightarrow \varphi(-x') = \varphi(x')$$

$$\Rightarrow A = 0$$

Cartesian Coordinates - Separation of Variables



$$\phi(x,y) = \sum_{i=0}^{\infty} X_i(x)\, Y_i(y)$$

$$= \sum_{i=1}^{\infty} \left[A_i \sinh \alpha_i x + B_i \cosh \alpha_i x\right] * \left[C_i \sin \alpha_i y + D_i \cos \alpha_i y\right] + \left[A_0 x + B_0\right] * \left[C_0 y + D_0\right]$$

$\phi(x,y) = \phi(-x,y) \Rightarrow A_i = 0$

$\phi(x,0) = 0 \Rightarrow D_i = 0$

$\phi(x,a) = 0 \Rightarrow \alpha_i = \frac{i\pi}{a}, \; C_0 = 0$

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{a} x \, \sin \frac{n\pi}{a} y$$

$$\phi\left(\tfrac{a}{2}, y\right) = V_0 \Rightarrow \boxed{\sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{2} \sin \frac{n\pi}{a} y = V_0}$$

Finding A_n

$$\int_0^a \sin\frac{m\pi}{a}y \, \sin\frac{n\pi}{a}y \, dy = \frac{a}{2}\delta_{mn}$$

$$\int_0^a \phi\left(\tfrac{a}{2}, y\right) \sin\frac{m\pi}{a}y \, dy = A_m \frac{a}{2}\cosh\frac{m\pi}{2}$$

$$\parallel$$

$$\int_0^a V_0 \cdot \sin\frac{m\pi}{a}y \, dy$$

$$\parallel$$

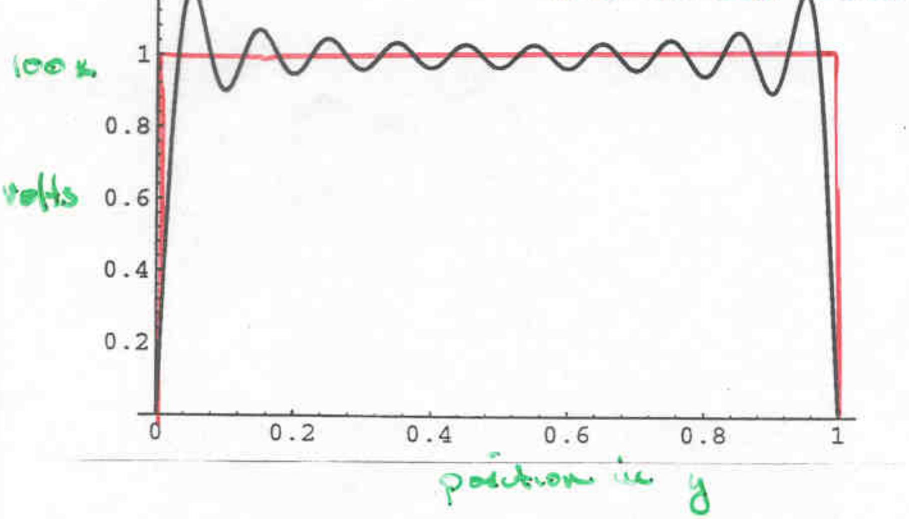
$$V_0 \begin{cases} \dfrac{2a}{m\pi} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} = A_m \frac{a}{2}\cosh\frac{m\pi}{2}$$

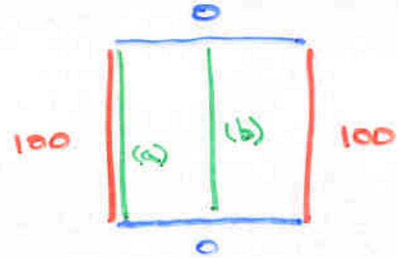
$$\Rightarrow A_m = \begin{cases} \dfrac{4V_0}{m\pi\cosh\frac{m\pi}{2}} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases}$$

$$\Rightarrow \boxed{\phi(x,y) = V_0 \sum_{m \text{ odd}} \frac{4}{m\pi} \frac{\cosh\frac{m\pi}{a}x}{\cosh\frac{m\pi}{2}} \sin\frac{m\pi}{a}y}$$

Square problem.

first 10 odd terms

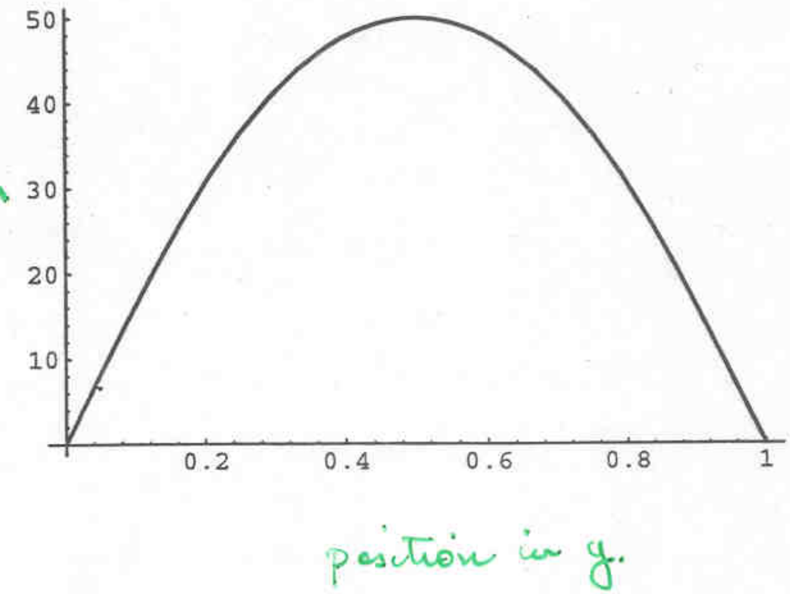


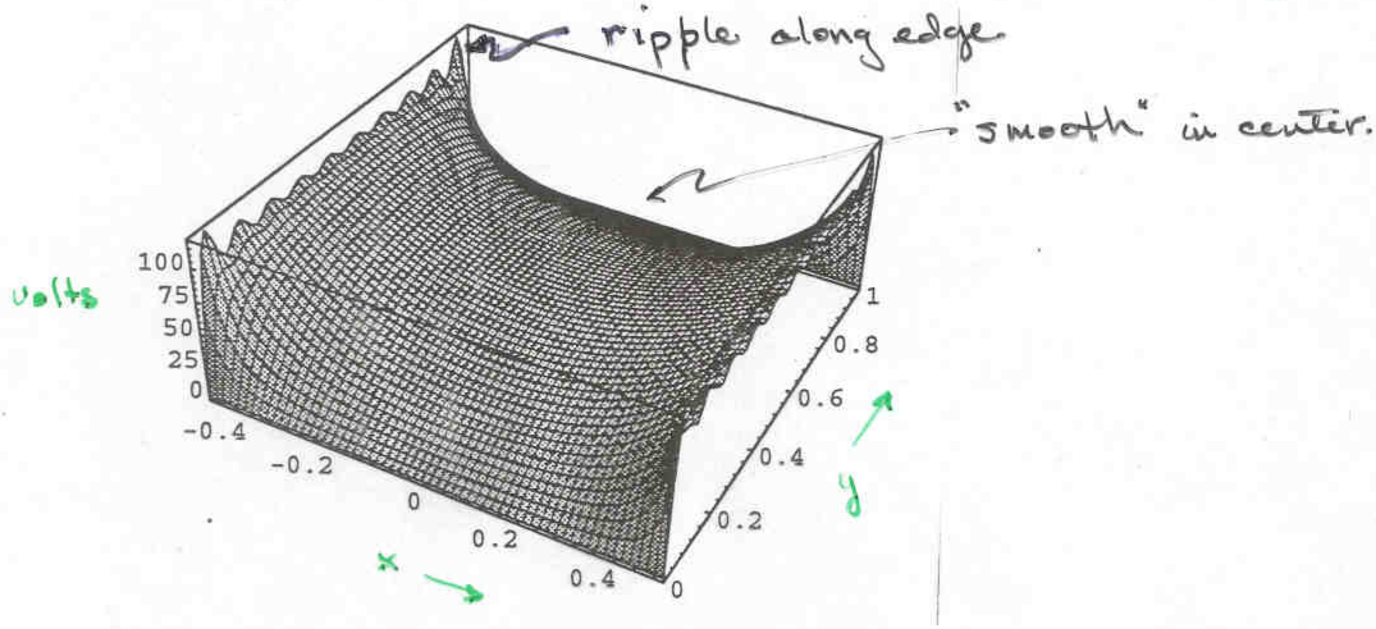


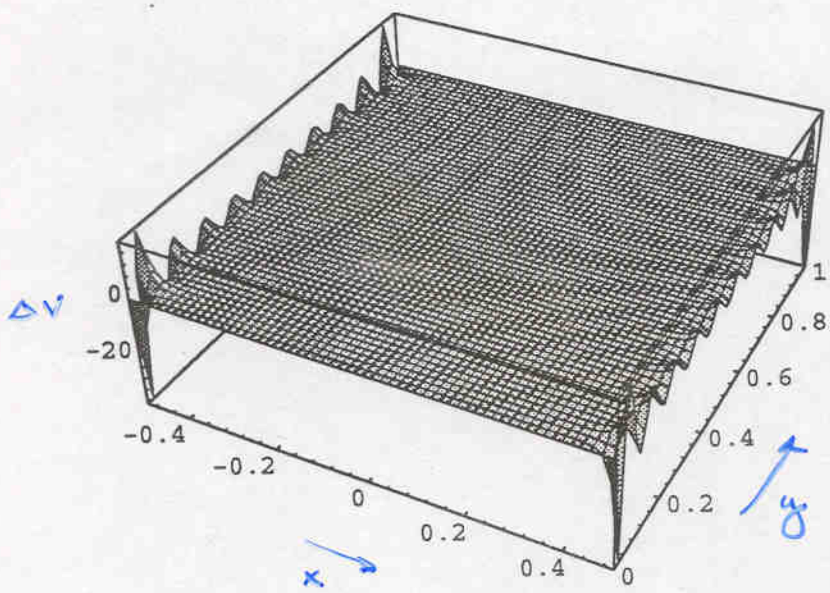
(a) voltage along edge

volts

(b) voltage through center







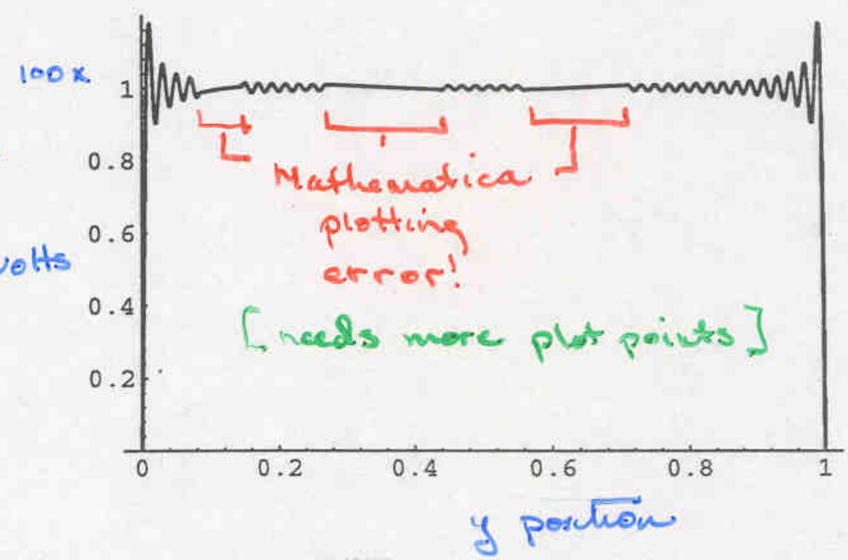
Σ first 10 odd terms

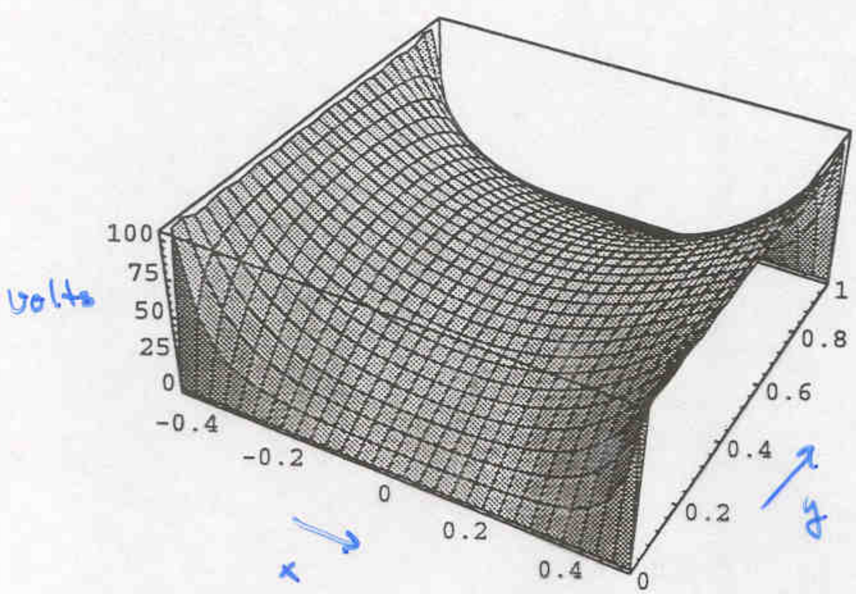
Theory − Σ

theory comes from conformal mapping.

Σ first 50 odd n

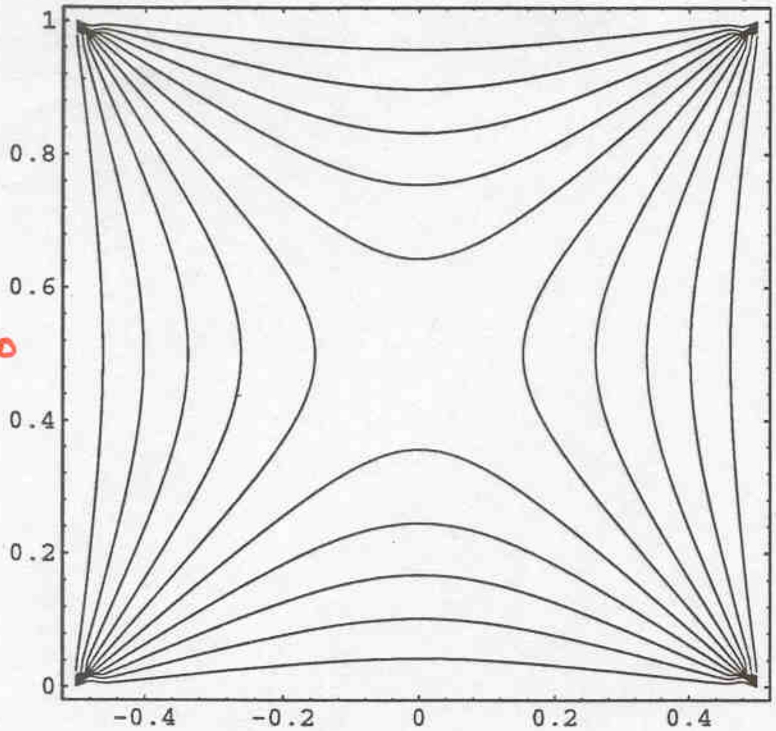
at (a)





Σ first 50 odd terms

$V=0$



$V=100$ $\qquad$ $V=100$

$V=0$

Equipotentials

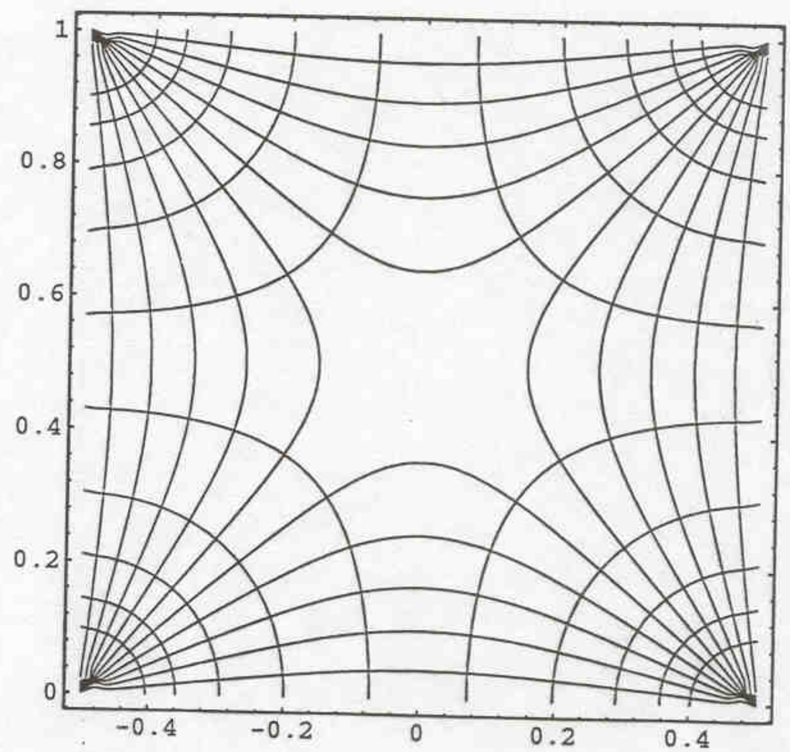
$$\phi(x,y) = \sum_{n=\text{odd}} \frac{4 V_0}{n\pi} \cdot \sin\frac{n\pi y}{a} \, \frac{\cosh\frac{n\pi x}{a}}{\cosh\frac{n\pi}{2}}$$

Sum over first 50 odd terms.

Field lines:

$$F(x,y) = \sum_{n=\text{odd}} \frac{4 V_0}{n\pi} \cdot \frac{\cos(n\pi y)-1}{2} \, \frac{\sinh n\pi x}{\cosh\frac{n\pi}{2}}$$

sum of first 50 odd terms.



2D Field Lines from Potential

Imaginary Part from Real Part

Cauchy - Riemann:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\hookrightarrow \quad V(x,y) = \int_0^y \frac{\partial U(x,y')}{\partial x}\, dy' + g(x)$$

Check Cauchy - Riemann:

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x} \qquad \frac{\partial V}{\partial x} = \int_0^y \frac{\partial^2 U(x,y')}{\partial x^2}\, dy' + \frac{\partial g(x)}{\partial x}$$

ok!

Note use of Laplace's equation here! $\Rightarrow$

$$= -\int_0^y \frac{\partial^2 U(x,y')}{\partial y'^2}\, dy' + \frac{\partial g(x)}{\partial x}$$

$$= -\frac{\partial U}{\partial y'}\Big|_0^y + \frac{\partial g(x)}{\partial x}$$

$$= -\frac{\partial U(x,y)}{\partial y} + \frac{\partial U(x,0)}{\partial y} + \frac{\partial g(x)}{\partial x}$$

If $g(x) = -\int_0^x \frac{\partial U(x',0)}{\partial y}\, dx'$ then ok

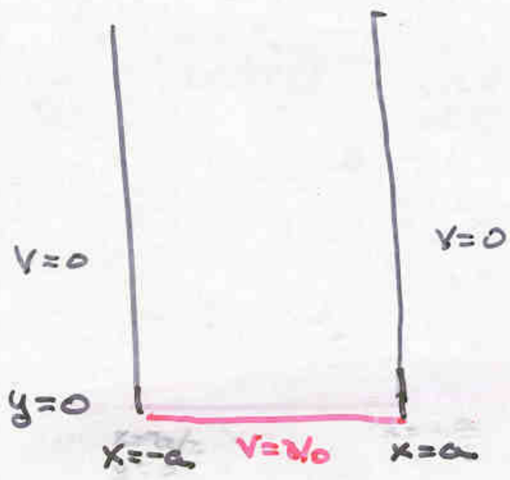
$$V'(x,y) = \int_0^y \frac{\partial U(x,y')}{\partial x}\, dy' - \int_0^x \frac{\partial U(x',0)}{\partial y}\, dx'$$

$$= V(x,y) - V(x,0) + V(x,0) - V(0,0)$$

$$= V(x,y) - V(0,0) \quad \text{renormalized } V!$$

It works!

STRIP BOUNDRY



$$\varphi(x,y) = \sum_{m=1}^{\infty} A_m \cos\frac{(2m-1)\pi x}{2a}\, e^{-\frac{(2m-1)\pi y}{2a}}$$

Considerations:

Symmetric $\Rightarrow$ Sine terms $= 0$

$B_0 x = 0$

No variation in y at sides $\Rightarrow D_0 y = 0$

Zero at $\{x, y\} = 0 \Rightarrow A_0, C_0 = 0$

Must go to zero as $y \to \infty \Rightarrow$ only negative exponent.

Zero at $x = \pm a \Rightarrow$ only odd cosines $\Rightarrow \alpha = \frac{(2m-1)\pi}{2a}$

Evaluate at $y=0$

$$\Rightarrow \quad a A_m = \int_{-a}^{a} V_0 \cos\frac{(2m-1)\pi x}{a}\, dx$$

$$= \frac{4V_0 a}{(2m-1)\pi}(-1)^{m+1}$$

$$\varphi(x,y) = \sum_{m=1}^{\infty} \frac{4V_0}{\pi}\frac{(-1)^{m+1}}{(2m-1)} \cos\frac{(2m-1)\pi x}{2a}\, e^{-\frac{(2m-1)\pi y}{2a}}$$